Understanding Structured Number Lines

Phase 1 Report July 2019

Report of an action research project

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Acknowledgements

We are grateful to the teachers involved in this research for working with us and helping us to explore structured number lines. The schools participating were:

Ilfracombe Infants and Nursery School (Ilfracombe)

Ladysmith Infant and Nursery School (Exeter)

Pilton Infants' School (Barnstaple)

Sticklepath Community School (Barnstaple)

Stoke Hill Junior School (Exeter)

The Castle Primary School (Tiverton)

Data processing and analysis was provided by Caitlin Gallagher

The project would not have been possible without funding and we are grateful for the funding support received from Devon County Council and the Jurassic Maths Hub

Summary of Findings and Recommendations

- Understanding of structured number lines does not appear as an explicit objective in the Primary National Curriculum. Whilst many teachers **use** number lines with children in primary mathematics classrooms, this research project showed that many children from across the primary age range do not understand the structure of the number line and their engagement with it has been limited to using it for counting rather than to understand the structure of the number system.
- Understanding of the structured number line as a measurement model is limited. This is shown by:
    - The difficulty children have with placing numbers on a number line, attending to the spaces between marks. Many children view the marks on a number line as somehow independent of and unrelated to the line itself and rely on counting in ones in an arbitrary way along a line in order to place or identify numbers.
    - The lack of understanding of moving from one mark to another, for example on a ruler measuring from 10 cm to 30 cm. The most common interpretation of this was that the measure was 30 cm in length.
- Too many children relied on either counting or a formal written method for both additions and subtractions that would be better solved mentally. Decisions in calculation need to be better informed by understanding of number and the number system including the position of numbers in relation to each other, application of known facts and understanding of the meaning of operation symbols (particularly subtraction symbol). This understanding needs to be explicitly connected as children engage with different parts of the number system such as decimals; it cannot be assumed they will make this connection automatically and independently.
- Understanding and connecting different ways to think about numbers and operations is essential if children are to develop a fluent, flexible approach to mathematics and in particular calculation. Not enough attention is currently given to making this connection nor to the importance of making informed decisions that take into account the numbers involved. Too many children are being expected to use a written method before they have secured understanding of the operations and of how to use this understanding to solve calculations.

    …there is growing evidence that once students have memorized and practiced procedures without understanding, they have difficulty learning to bring meaning to their work Russell 2000

Introduction

This project was set up in response to growing evidence that pupils in England are not developing an understanding of the structured number line and that this has implications with regard to understanding mathematics across several domains of the national curriculum. A key source of evidence for this has been the IMAPs project.

Investigating Mathematical Attainment and Progress: The Low Attainment in Year 9 Project (IMAP) is a collaboration between the University of Nottingham and colleagues at Durham University (Professor Robert Coe, Professor Steve Higgins, Dr Lee Copping) and King's College London (Professor Margaret Brown), led by Professor Jeremy Hodgen, funded by the Nuffield Foundation and running from September 2015 to April 2019 (final report pending). This project has explored a number of questions including the following:

> *What mathematics do low attaining secondary students understand, and what are their particular strengths and weaknesses in number, multiplicative reasoning and algebra?*
>
> *To what extent do low attaining students' prior understandings of mathematics, and of particular mathematical topics, help to explain the existence of the attainment gap?*

One of the findings of the IMAPs project is that pupils in KS3 who struggle with maths do **not** understand **number lines as representations of the number system** and this has been identified as a key area for development.

We were interested in exploring how primary pupils understand the structured number line and how we can support them to understand it so that this understanding has an impact on understanding in other areas of mathematics, in particular calculation.

Structured number lines

> *Number lines with marked line segments are referred to as structured number lines…number sequencing on a number line goes beyond knowing the order of number names. Because the number line is a measurement model, rather than a counting model, numbers on the number line are representations of lengths rather than simply the points they label (Fuson, 1984). Thus, in determining an unknown marked position on a number line, the proximity of the unknown from the known numbers is important. For example, on a number line that commences with 0 and concludes with 10, the number corresponding to a marked position between these numbers depends on its proximity to each of the numbers.*
>
> Diezmann and Lowrie 2006

There is an important distinction to make between structured number lines and empty number lines. Arising from their use in the Netherlands, empty number lines emerged in England, as a model for supporting children to calculate; they are a fundamentally different model to the structured number line as they are not built on notions of measurement.

This project was **NOT** focussed on using number lines to '**do**' things such as calculate, and was not about the empty number line. It was focussed on understanding the structured number lines as a way to understand the structure of the number system. In turn, it was then intended and anticipated that this would support understanding in other areas, specifically measurement, statistics and calculation.

The project focussed on the following research questions:

- What are the key elements to understanding number lines as representations of the number system?
- Which activities help support this understanding?
- What is the impact of understanding number lines?

The project team identified three related elements that were the focus of both professional development sessions and teaching materials:

- Understanding that number lines have regular spacing and that the spaces relate to a set unit.
- Exploring points between marks
- Making decisions about where to place numbers and how to identify numbers in relation to marked numbers

Overview of action research project

Project sample

This project involved six schools from across Devon; three infant schools, two primary schools and one junior school. Each school identified one or two lead teachers and selected the classes that would be involved in the pre- and post-project assessments. Teachers from across each school participated in the three professional development sessions, which were school-based, and accessed the accompanying materials. One school was delayed in joining the project.

School	Assessment	Teachers involved
Ilfracombe Infants and Nursery School	Y1: 12 children from 4 classes Y2: 4 classes	EYFS to Y2
Ladysmith Infant and Nursery School, Exeter	Y1: 9 children from 3 classes Y2: 3 classes	EYFS to Y2
Pilton Infants' School, Barnstaple	Y1: 12 children from 2 classes Y2: 2 classes	EYFS to Y2
Sticklepath Community School, Barnstaple	Y1: 6 children from 1 class Y2: 1 class Y3: 1 class Y4: 1 class Y5: 1 class	EYFS to Y6
Stoke Hill Junior School, Exeter	Y3: 1 class and 6 children from 2 classes Y4: 1 class and 6 children from 2 classes Y5: 1 class and 6 children from 2 classes Y6: 1 class and 6 children from 2 classes	Y3 to Y6
The Castle Primary School, Tiverton	Y2: 1 class Y3/Y4: 1 class Y5/6: 1 class	EYFS to Y6

Due to the fact that half of the sample were infant schools and the youngest children where whole class assessments were undertaken were in Y2, there was substantially more Y2 data collected than any other year group:

Year Group	Number of Children
1	43
2	251
3	70
4	69
5	64
6	47

Project structure

The project was supported by two maths advisers from Babcock LDP acting as lead researchers for the project. The project was part of a focus on improving outcomes for all children, a priority for Devon with funding from Devon County Council. It also received funding as an innovation project for the Jurassic maths hub.

The structure of activities was as follows:

Activity	Details	Half term
Data collection	Assessments undertaken by lead researchers	Autumn 2
PD 1	Led by lead researchers with accompanying materials provided for teachers	Autumn 2
PD 2	Led by lead researchers with accompanying materials provided for teachers + teacher feedback	Spring 1
PD 3	Led by lead researchers with accompanying materials provided for teachers + teacher feedback	Spring 2
Collaborative Lesson Research cycle	Lead researchers with lead teachers from five of the project schools worked together as a planning team with a live lesson observed by additional teachers and maths professionals (see appendix 1)	Spring 2 and Summer 1
Data collection	Assessments undertaken by lead researchers	Summer 1
Feedback meeting and online survey	Teacher feedback	Summer 1

The project activities took place in each individual school except the collaborative lesson research cycle which brought teachers from five of the schools together as members of the planning team located in one of the project schools.

Data collection

- Lead researchers collected the data
- Short assessments (around 20 minutes):
    - For Y1 selected focus children (pupil premium where appropriate) in trios
    - From Y2 onwards whole class and some selected focus children (pupil premium where appropriate) in trios

	Question	Numbers	Understanding assessed
1	• Can you mark these numbers on the number line (horizontal) • Mark a number of your own choice which is easy to place • Can you mark these numbers on the number line (vertical) Say and write up the numbers each time	**Y1** Number lines 0 to 20, tens marked and numbered Numbers: 9, 5, 11	• Equal spacing • Position numbers relative to each other • Number lines in different orientations
		Y2 Number lines 0 to 60, tens marked and numbered Numbers: 35, 29, 51 Numbers: 45, 39, 21	
		Y3 Number lines 0 to 60, tens marked and numbered Numbers: 35, 29, 51 Numbers: 45, 39, 21	
		Y4 Number lines 100 to 500, hundreds marked and numbered Numbers: 350, 290 410 Numbers: 210, 390, 450	
		Y5 Number lines 4100 to 4500, hundreds marked and numbered Numbers: 4350, 4290, 4410 Numbers: 4210, 4390, 4450	
		Y6 Number lines 0 to 6, ones marked and numbered Numbers: 3.5, 2.9, 5.1 Numbers: 2.1, 3.9, 4.5	

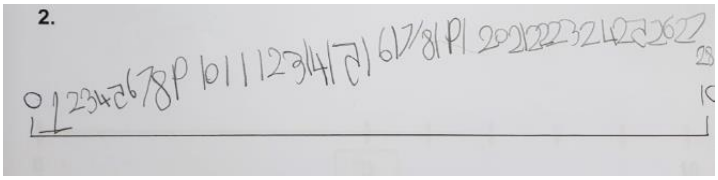
	Question	Numbers	• Understanding assessed
2	Draw a number line from X to Y and mark numbers Say and write up	**Y1** 0 to 10, all counting numbers marked	• Equal spacing • Number line as measurement model
		Y2 0 to 10, all counting numbers marked	
		Y3 0 to 100, mark the tens	
		Y4 100 to 200, mark the tens	
		Y5 2500 to 2600, mark the tens	
		Y6 2500 to 2600, mark the tens	
3	This is a piece of ribbon that has two colours. How long is the red part of the ribbon?	**Y1** N/A	• Number line as measurement model • Moving from mark to mark
		Y2 red goes from 10 to 30	
		Y3 red goes from 100 to 300	
		Y4 red goes from 1000 to 3000	
		Y5 red goes from 0.1 to 0.3	
		Y6 red goes from 0.1 to 0.3	
4	What number goes here? Five options offered to choose from – bold indicates the correct number	**Y1** 0 to 10 with 5, 6, 7, 8, 9 marked (not numbered) Options: 1, 2, **5**, 4, 8	• Number line as measurement model • Midpoints
		Y2 0 to 20 with 5 and 10 marked (not numbered) Options: 18, 1, 10, **5,** 2	
		Y3 0 to 200 with 100 and 150 marked (not numbered) Options: 190, 2, 50, **150**, 75	
		Y4 200 to 400 with 300 and 350 marked (not numbered) Options: 300, **350**, 220, 390, 202	
		Y5 1200 to 1400 with 1300 and 1350 marked (not numbered) Options: 1300, 1**350**, 1220, 1390, 1202	
		Y6 1200 to 1400 with 1300 and 1350 marked (not numbered) Options: 1300, 1**350**, 1220, 1390, 1202	

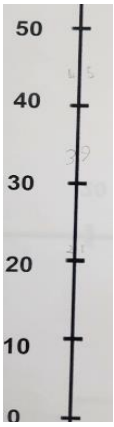
	Question	Numbers	Understanding assessed
5	What's this number? Mark…on this line. Say and write the number that has to be marked	**Y1** 0 to 20, ten marked (not numbered) 5 marked with an arrow, mark 13	
		Y2 0 to 50, tens marked (not numbered) 35 marked with an arrow, mark 23	
		Y3 50 to 100, tens marked (not numbered) 85 marked with an arrow, mark 63	
		Y4 0 to 500, hundreds marked (not numbered) 350 marked with an arrow, mark 230	
		Y5 5**0**0 to 1000, hundreds marked (not numbered) 850 marked with an arrow, mark 630	
		Y6 15**0**0 to 2000, hundreds marked (not numbered) 1850 marked with an arrow, mark 1630	
6	Calculations	**Y1** 11 – 9, 8 + 3	• Applying known facts and understanding of numbers to add a single digit number • Choosing to solve subtraction by focusing on the gap between the numbers
		Y2 8 + 2, 3 + 3, 52 – 48, 37 + 6	
		Y3 8 + 2, 3 + 3, 52 – 48, 37 + 6	
		Y4 8 + 2, 3 + 3, 552 – 548, 837 + 6	
		Y5 8 + 2, 3 + 3, 2002 – 1998, 3837 + 6	
		Y2 8 + 2, 3 + 3, 5.2 – 4.8, 3.7 + 0.6	

Assessment observations

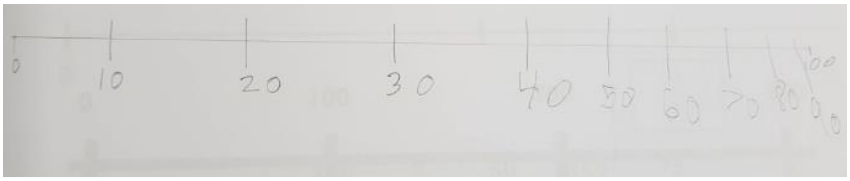
The children's responses to the assessments were shared with each school at the start of the first PD session and staff were asked to discuss and share what they noticed. Across the schools the same things were observed:

- The children did not have an understanding of the number line as a measurement model (Fuson 1984). This included a lack of understanding of:
    - The marks on a number line and how to mark numbers on a number line

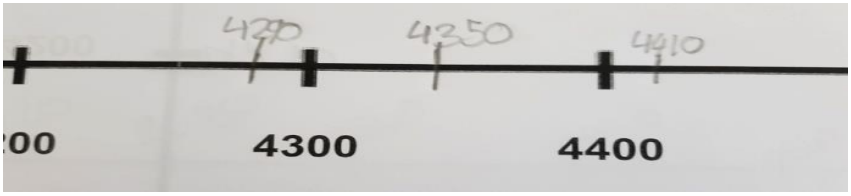




    - Equal spacing

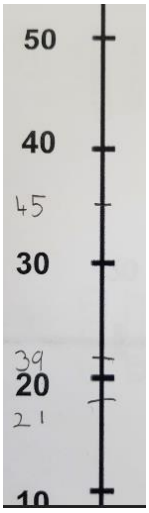


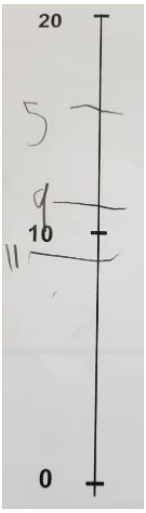
    - Where to place numbers in relation to numbers already placed on the line including using midpoints as a reference



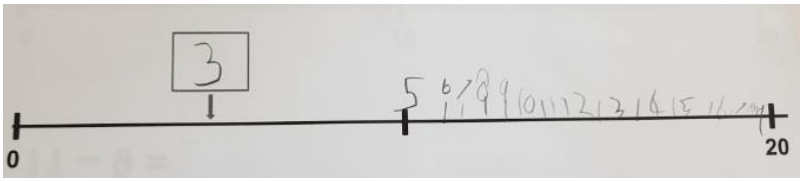
    - Moving from mark to mark

- Better understanding of horizontal lines than vertical number lines

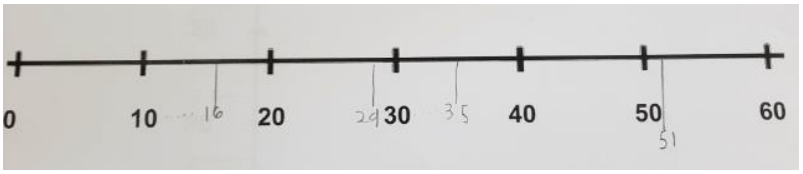




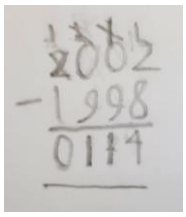
- Reliance on counting imaginary marks and ignoring existing marks and numbers when placing numbers.

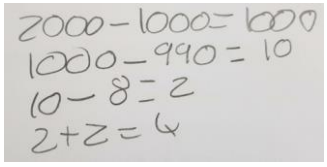


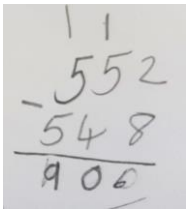
- Better at marking and identifying mid-points and knowing numbers close to each other.

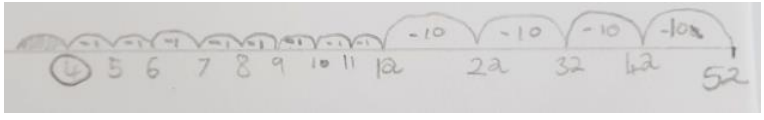


- Not choosing to focus on the gap between the numbers for the subtraction; poor choice of method, often resulting in incorrect responses.

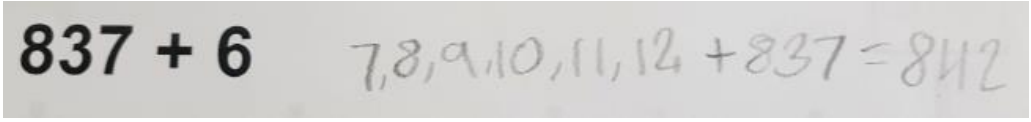


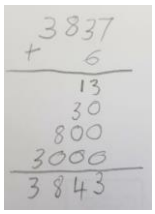


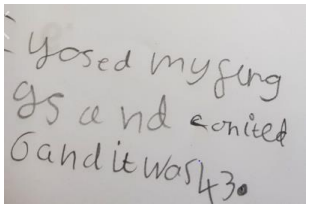


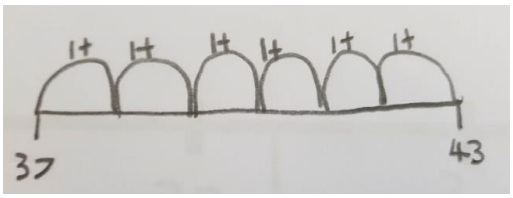


- Not choosing to partition 6 into 3 and 3 to add it to a number ending in 7, choosing other methods e.g. counting on in ones or column addition









The research team had not anticipated that many children would not understand how to make a mark on a number line to indicate the position of a number; this was added into the materials in response to the assessments.

Professional Development Sessions and Teaching Materials

Each PD session explored the mathematics in the teaching materials for a particular aspect of understanding number lines for EYFS to Y6 as appropriate:

- Session 1: Understanding that number lines have regular spacing and that the spaces relate to a set unit.
- Session 2: Making marks, moving between marks and exploring mid-points between marks on a number line
- Session 3: Making decisions about: where to place numbers, how to identify numbers and how to apply understanding of the position of numbers when calculating

The materials provided were journeys through different aspects of mathematics fitting with the three sessions above. The intention was that teachers used them with pupils in their classes as necessary; they could be used for regular, short, number line sessions, as warm ups or cool downs in maths lessons or as the focus of a teaching sequence. The aim throughout was to support all children to understand so it was for teachers to determine how much they needed to use the activities in order for understanding to be secured.

Teachers were asked to reflect on the activities between the PD sessions, using the following questions as prompts:

- Which bits did the children struggle to understand or find ‘tricky’?
- Which activities worked well? Why?
- Which activities need adjusting? How?
- Is anything else needed?
- Are we focussing on the right things?

Collaborative Lesson Research Cycle

> *For Japanese educators, lesson study is like air, felt everywhere because it is implemented in everyday school activities, and so natural that it can be difficult to identify its critical and important features.* T Fujii 2014

Collaborative Lesson Research is the Japanese form of Lesson Study using live lessons to answer shared questions about teaching and learning. We define Collaborative Lesson Research (CLR) as having the following components:

- A clear research purpose
- Kyouzai kenkyuu
- A written research proposal
- A live research lesson and discussion
- Knowledgeable others
- Sharing of results

The model for the CLR cycle was based closely on the Japanese model with substantial time spent by the planning team exploring the mathematics and agreeing the lesson proposal. We chose to explore the application of understanding of numbers in the number system to decisions made when subtracting, as the results from the pre-project assessments showed that across all schools the children were generally not making good decisions about how to solve a subtraction where the numbers involved were close together. The full lesson proposal can be found in appendix one.

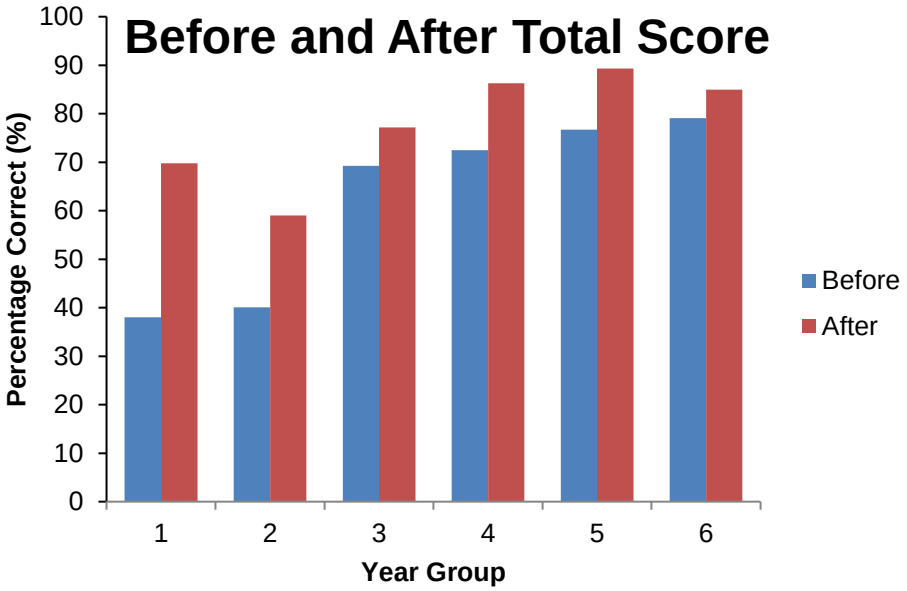
Feedback from teachers

Forty-four teachers from across the six schools completed an online questionnaire which included the following questions:

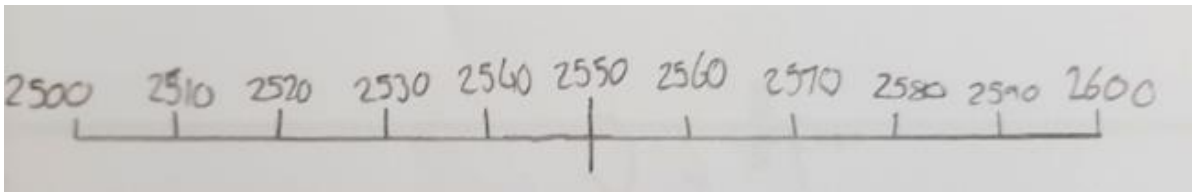
- How much have you used the ideas in the materials?
- Which activities have you used?
- What has worked well? Why? What else is needed? Why?
- What impact have you noticed? Have there been any surprises?
- What are you taking from the project?

Phase 1 Research Findings

The assessment data shows that all year groups improved with the biggest change in Y1, where the assessments at the start of the project showed the lowest scores.



The biggest improvement was in question 2, which was focussed on understanding of equal spacing.



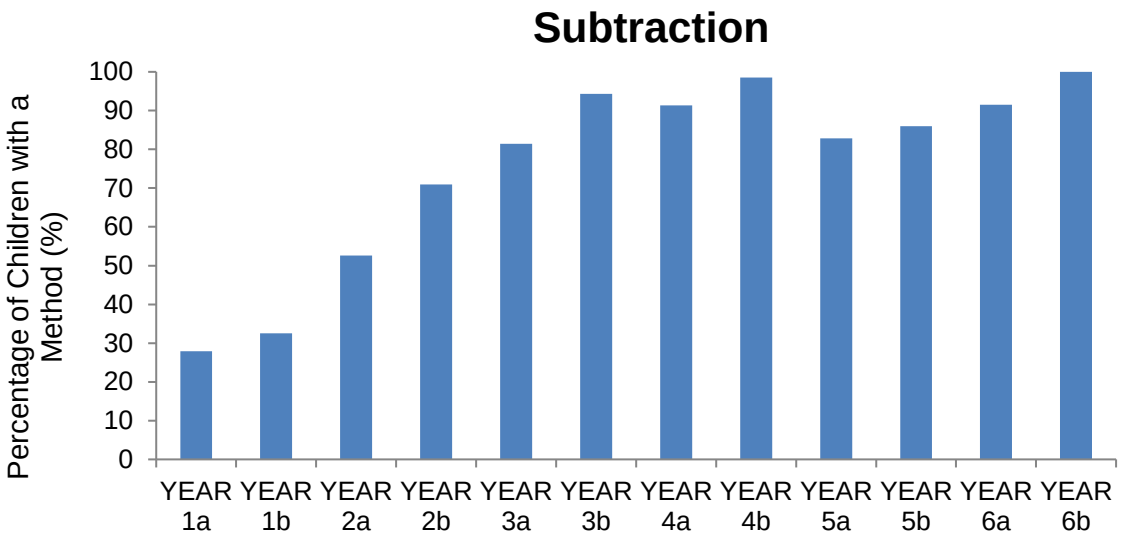
It was clear from the teacher feedback that there had been a greater focus on using the materials related to the mathematics in PD1 particularly looking at equal spacing and both horizontal and vertical lines. Teachers across the age groups realised that they had made assumptions about children's understanding of the strucutre of number lines and that this needed to be explored. They made observations of the learners that then meant they changed their practice:

- Equal spaces as steps was a key piece of understanding that we often revisited and hooked back to before building on the new number line skill…We loved the activities where the children discuss "Is it equal spacing? Is it not?" Which is the odd one out?" as they were then reasoning about the equal steps.
- The importance of taking time to explain how number lines actually work rather than ploughing straight in to using them.
- The first session looking at equal spacing the children really engaged with it and were very vocal with their suggestions and ideas. As the sessions have moved through the children

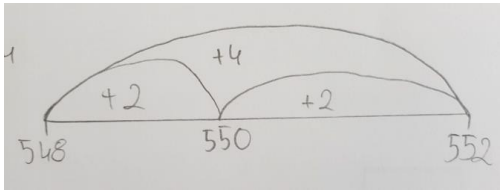
have remembered aspects from the previous session and always remember that the number lines need equal spacing

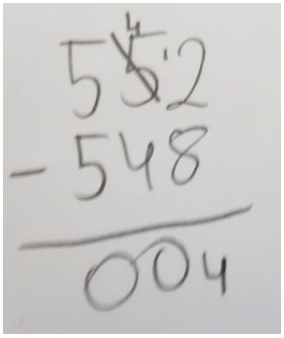
- The change in equal spacing has been good. Drawing number lines has worked well. To be honest the first 2 packs we did in more detail and therefore the children have improved.
- In year 1 we did a lot around equal spacing and equal steps. We focused on the steps in between numbers rather than jumps. We also exposed the children to various number lines, both vertical and horizontal.
- I am going to continue to incorporate number lines into my teaching. I have realised the effectiveness of them. I have also learnt the importance of teaching and exposing both vertical and horizontal number lines to the children.
- Reminding children to find the mid-point has worked well. Also focusing on equal steps for drawing scales in data handling and recently in lessons on capacity so working out the scale on measuring cylinders, jugs etc. for practical measuring.
- It was interesting to show a variety of number lines with different spacing. The children immediately noticed which ones were not evenly spaced. Stepping out the numbers and saying the number worked well as the children were able to understand that even if the numbers were hidden, they still knew the value of each step.

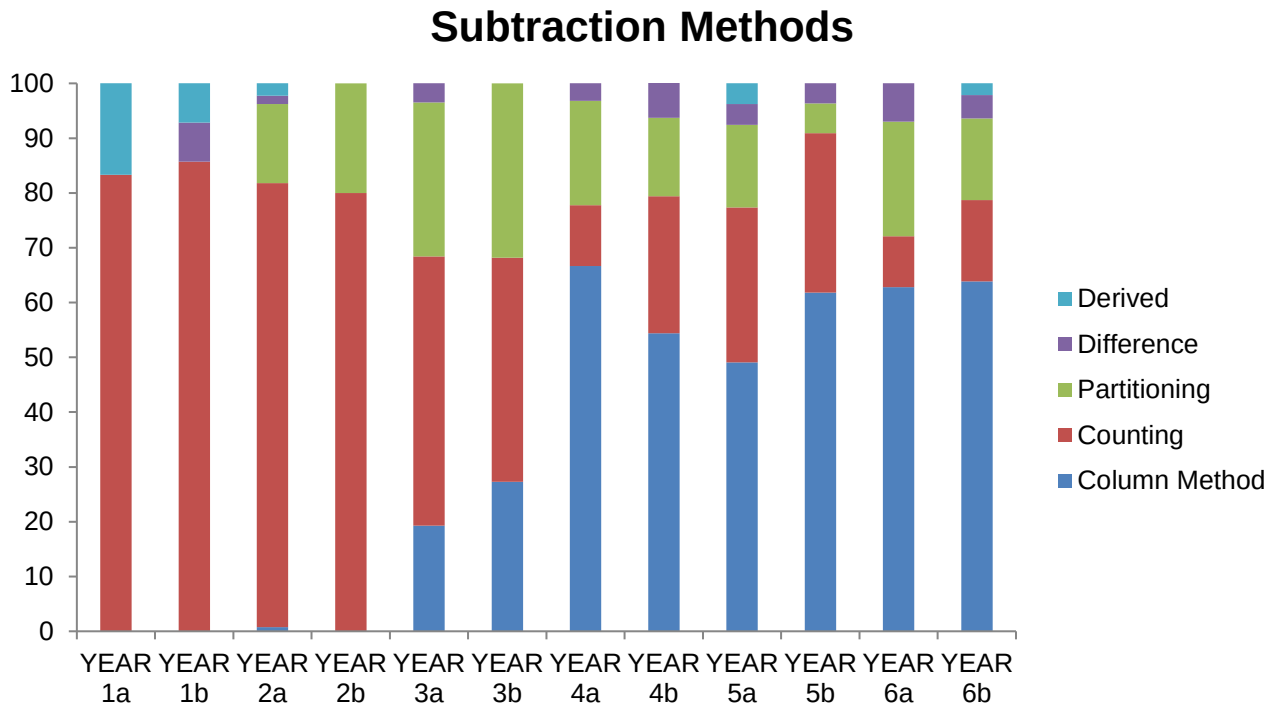
However, the impact has been limited to understanding number lines and has not, as yet, had the impact on calculation that was hoped for. The chosen method was known for most of the children in KS2.



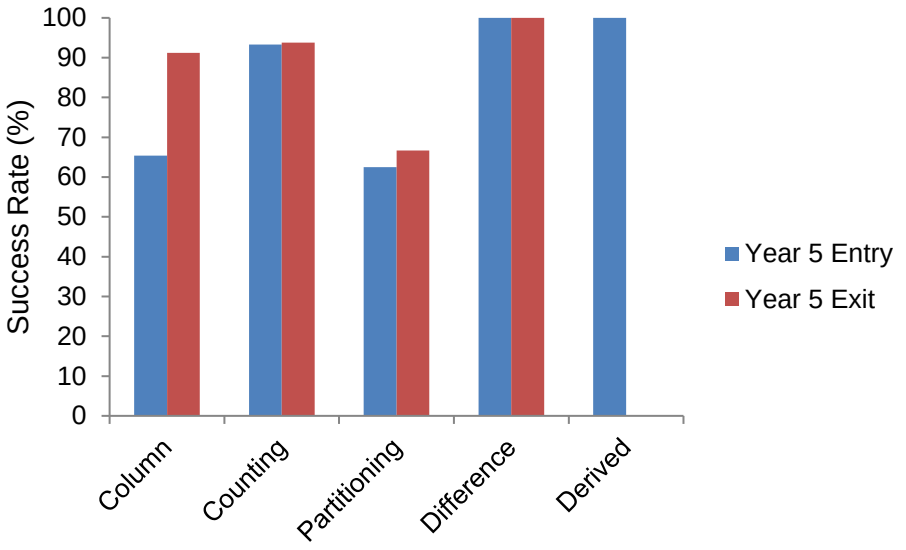
The percentage of children choosing to solve a subtraction where the numbers are close together using a formal written method increased from Y3 to Y6 in both the pre- and post- assessments and the project did not have a signifcant impact on the decisions made by the children; in fact there was an increase in the percentage of children choosing the formal written method in the post-project assessments in Y3, Y5 and Y6. Only in Y4 was there a reduction in the percentage of children using the formal written method for the subtraction.

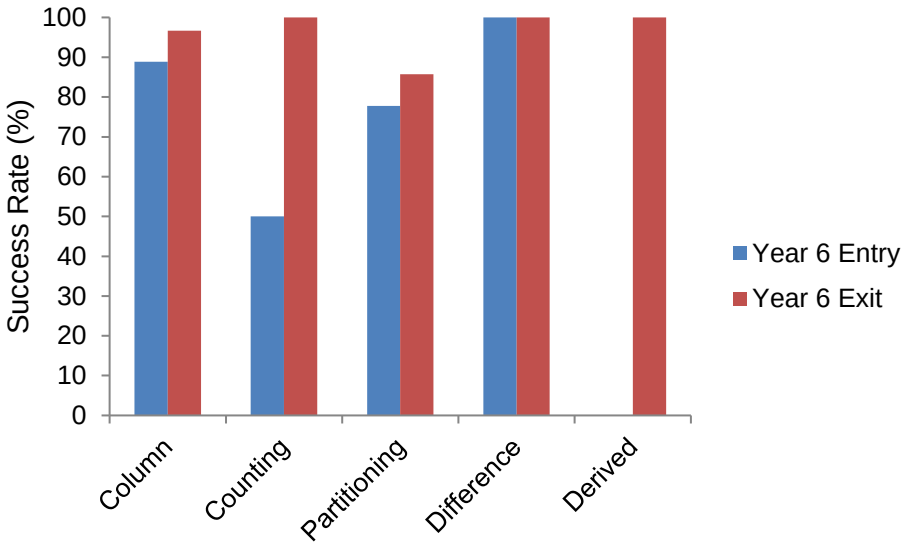






In Y5 and Y6 children choosing to find the difference were 100% successful.





There are a number of factors to consider in relation to this:

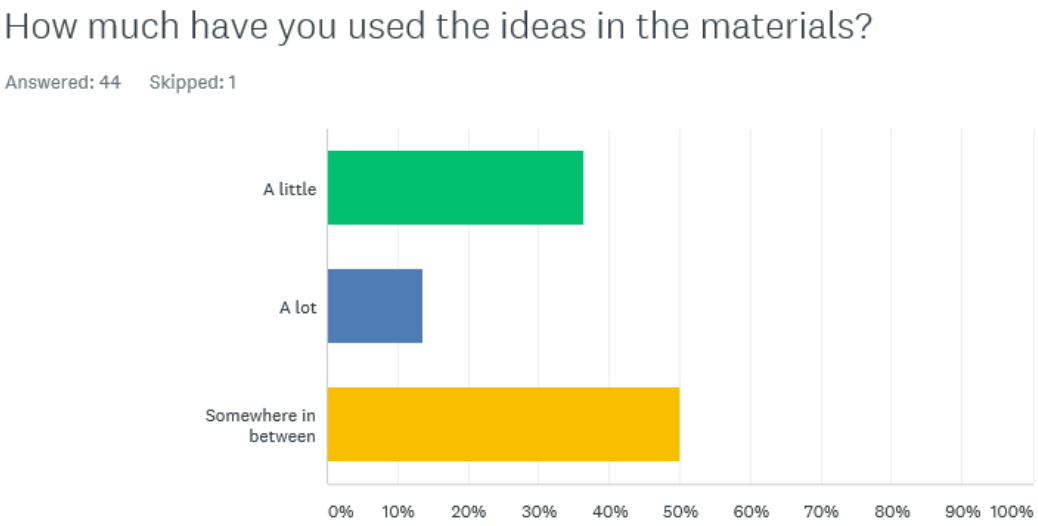
- Teachers struggled with appreciating the need for children to symbolically represent finding the difference as a subtraction if the method used to solve the problem was an addition.
  There is a daunting range of situations in which we have to learn to recognize that the appropriate operation is subtraction…Each of these has its own characteristic language patterns, all of which have to be connected in the learner's mind with subtraction.

  Haylock and Manning p.91

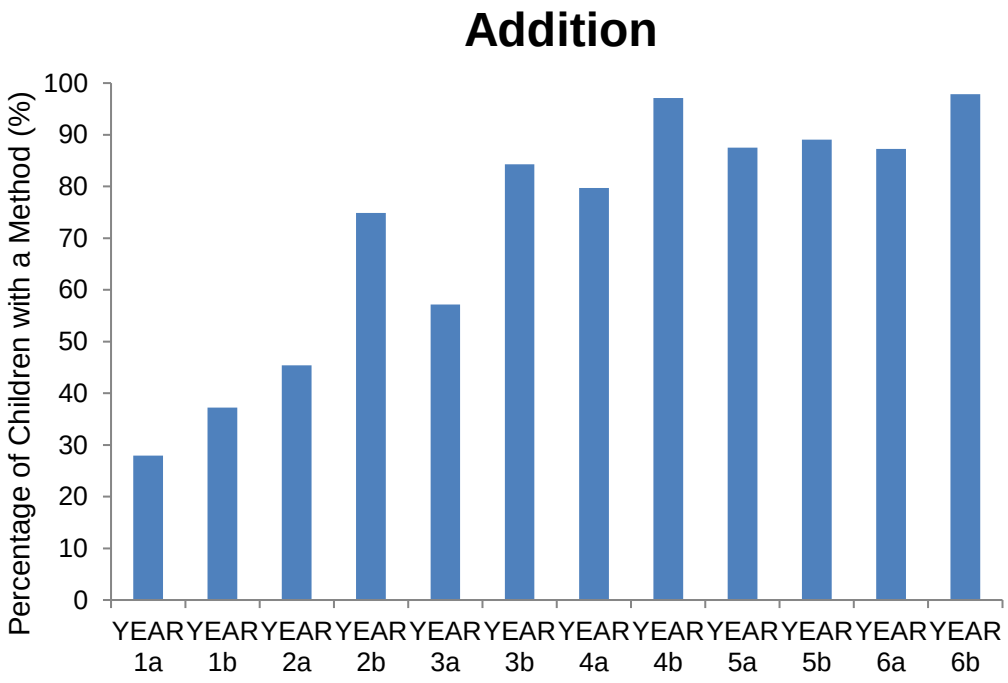
  For example finding the difference between £3.80 and £3.50 was recorded as £3.50 + 30p = £3.80 because this reflected the method. There was a lot of exploration of the need to also record the calculation that was being solved as £3.80 - £3.50 so that children connected the idea of adding on to find the difference to subtraction so that when faced with a subtraction they

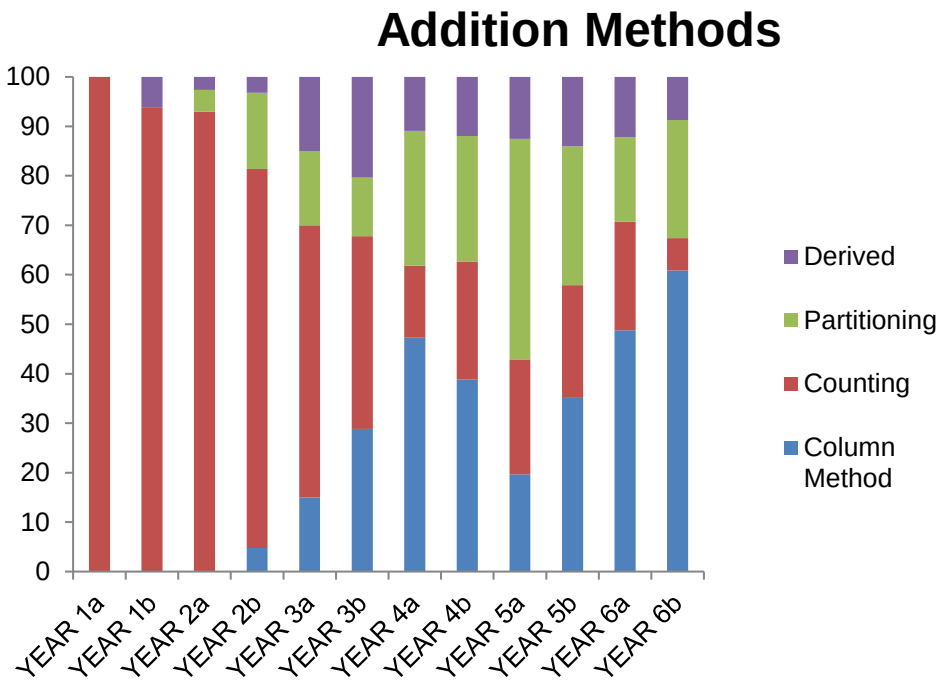
considered this as a potential method. The CLR cycle explored this in detail and exposed the complexities.

- Teachers made more use of the early materials, associated with PD1, and limited use of the materials associated with PD3. Overall, the use of the materials was variable, with only a small percentage of the teachers making a lot of use of them:



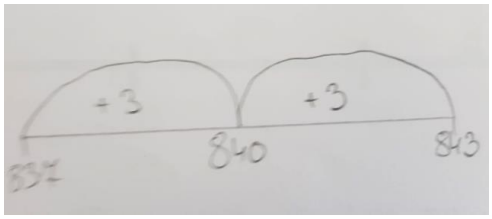
Again, where the method was known for the addition, which involved adding 6 to a number that ended with 7 in all year groups from Y2 to Y5 (in Y6 it was adding six tenths to a number ending with seven tenths) there was again an increasing use of a formal written method for what should have been a mental calculation from Y3 to Y4 and Y4 to Y6, with Y5 slightly lower.

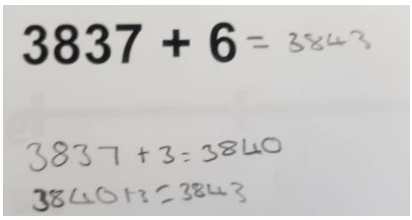




There are implications about decision making and making connections between known facts and different parts of the number system, as the Y6 question involved decimal numbers.

Examples of children's explanation of their mental calculation:





As well as the assessment data, we have the observations of the teachers who considered the impact on their children, which included the following:

- Children more confident in their understanding of number lines- able to point out when they are incorrect and starting to be able to explain how they know it is incorrect. Still struggle to draw no lines freehand. It may be useful to include more context examples as all struggled with the ribbon example.
- Quality of conversation much better and more confidence expressing opinion- Greater understanding of equal spacing really helped understanding in multiplication and division
- The children have developed their understanding about equal spacing and equal steps. The children can now recognise when a number line has used equal spacing or not.
- More confident equal spacing. More automatic use of midpoint. More confident language of talking about it.

- Some children see recognise that some large numbers, which are close together can be subtracted without formal calculations
- It had a good impact on the children and their knowledge of the number line and it was good to see them apply the skills they had learnt, using a number line with missing numbers.
- Children are more confident using number lines, are more likely to answer missing numbers correctly. Children have realised that number lines don't have to be straight.
- Children enjoyed the taking steps part and enjoyed taking the physical steps and relating this to the more abstract idea of a number line.
- Children explaining their thinking. They have better understanding about number and the meaning of numbers. All children were able to access it and there has been progress with the SEN children as well as the other children in the class. Next steps- working out subtraction number sentences thinking about the numbers being close together.
- I have been surprised how it has supported all of the children in my class. I think that all children have benefited from this. The children have been able to explain their answers with clarity and understanding to demonstrate their answer using the number line.

Impact on participants' professional learning

It is clear from teachers' feedback that many of them have a renewed appreciation of the importance of understanding the structured number line and the potential for this to support understanding of both the number system and calculation. There was particular interest in and understanding of the equal spacing on a structured number line.

- We have spent a lot of time drawing numbers over the last few years using sticks and dots. Children can talk about the parts of a number and what it represents but we have lost the connections between numbers which number lines visually support. I was surprised that the children had no awareness of equal spaces so the importance of explicitly teaching this discrete knowledge is essential.
- Who knew how much there is to think about in number lines! Really helped me to understand the different steps involved in understanding what goes drawing and using number lines.
- It has been brilliant and has really helped build an understanding of number. I love that it was so appropriate for EYFS as often CPD starts at Year 1. Thank you for this amazing CPD.
- I think this research project has been helpful in getting me to think about the complexities of number lines.
- I was surprised that equal spacing between numbers is still difficult for some in my class, especially in different contexts like drawing graphs.

- The importance of visual representations of numbers so they are not just an abstract concept, that children need to see numbers represented like this to understand them and what they represent
- Use of number lines in across the math curriculum instead of just place value. Use of number lines in context or not horizontal.
- I realised how much I have certain pre-conceptions of my own and how I will automatically set certain activities/pictures up without thinking.

The CLR cycle as a model of professional development was very well received especially by those involved in the planning team. This was where the most professional learning took place during the project:

- It was a very informative experience to think in detail about every part of a lesson which I do on a daily basis. The impact of subconscious decisions is huge and has allowed me to reflect on the impact it has on my own practice. It was great to see through the whole process and see how the live lesson reflected what we had planned.

- The process of planning successful maths lessons is very complex and requires careful thought about many aspects. However, the lesson itself will also sometimes be quite fluid and may evolve in a different way to the initial plan as the children's understanding/misconceptions are revealed. Discussion is a very important part of successful planning and it is key to focus on the language being modelled and the methods that we hope the children will be able to use over time. Those who have a good understanding of number need to be challenged to perfect their explanations regarding their methods and understanding of how they have reached their conclusions.

- The benefits of using research to inform professional practice were apparent in this process, e.g. research related to different addition and subtraction structures and research reflecting how children find it challenging to use indirect addition when finding the difference*

*See appendix 1 for further information and feedback on this experience.

The project involved the whole staff in each school so this has aided conversations across the school. Many of these have focussed on how number lines need to be built into the maths planning/curriculum in the future, at all ages:

- We need to go back to our long term maths plans, particularly place value and build in the development of number lines into our current sequences. I feel like I used to use number lines regularly (5-10 years ago) but somewhere along the way we have got more focused on pictorial images.

- The impact of going back to basics and working your way up with number lines. Helpful to do this each year to then adapt for the new number progression of year groups.
- I think that I would start right back to basics when starting the new school year in order to build on the number line knowledge (spacing, half way/quarter way, drawing your own, plotting numbers that aren't on a mark) and not assuming that they understand them. I will also be bringing number lines across the different units more, using them in to help with fluency.

Limitations and challenges

There were several limitations and challenges with this research project including the following:

- Assessments
  - These were devised by the research team. There was no external analysis or testing of the assessments.
  - The questions were focussed on understanding anticipated as key; they over-emphasised certain aspects of the mathematics and excluded other elements.
  - The assessments with Y1 children were undertaken in trios and this enabled the researchers to probe understanding. For the Y2 to Y6 assessments, which were primarily done as a whole class, it was not possible to probe understanding beyond what the children chose to communicate on their papers, and there was evidence of some children copying from each other, so the data is limited.
- Materials and teaching time
  - There were no expectations placed on participating teachers with regard to the number of sessions they would run with the materials or the number of activities they would use. This led to more use of the early materials and some teachers using them for just one or two sessions in the year.
  - The materials were offered as a sequence of activities and proved difficult for the teachers to incorporate in their maths planning. More guidance was needed about how to use them linked to different domains in the national curriculum.
  - The materials followed the journey of the mathematics and a single activity could run over many sessions. Some teachers found these hard to access because of this and as a consequence limited their use of them.

- Calculation
    - Understanding structured number lines can contribute to decision making in calculation but there are other factors that also need to be considered in order to have the impact desired. In the case of subtraction, this includes understanding the different ways you can choose to think about a subtraction.

Conclusion

The research project has been incredibly revealing. It has had an impact on teachers' thinking about structured number lines and children's understanding of them. The link to making decisions when calculating has not yet been realised and is a much longer term goal. The focus taken for the CLR cycle emerged from the project and existing research and was challenging for all involved to fully appreciate. We have realised just how important this is, which we had not anticipated. There is potential for this work to influence understanding of measures and data as well as number and calculation. The next step is for us to run a second phase of the project, with the materials more clearly linked to the different domains in the curriculum and presented in a more accessible way. We have offered schools in Devon the opportunity to participate in phase two and look forward to building on this work.

Phase 2

Phase 2 will build on the findings from phase 1, with changes as follows:

- Refining and linking materials to the national curriculum
- Professional development will be two whole days
    - Taking place in the autumn term
    - Focussed on understanding the mathematics and how it links to the national curriculum
    - Identifying opportunities to include use of the materials
    - Exploring implications for calculation early in the year so that there is time to work on this
- Separated notes relating to the lesson flow and teacher guidance
- Activities simplified so that each focuses on one key idea
- Assessments undertaken one to one in order to allow assessors to probe understanding
- Assessment materials refined and externally viewed
- Participating teachers will be expected to use the materials in at least 20 sessions during the year

References

Diezmann T. and Lowrie C. M. 2006 Primary Students' Knowledge of and Errors on Number Lines: Developing an Evidence Base

Fujii T. 2014 Implementing Japanese Lesson Study in Foreign Countries: Misconceptions Revealed *Mathematics Teacher Education and Development* 2014, Vol. 16.1, 65–83

Haylock, D. and Manning, R*. (2014). Mathematics explained for primary teachers. 5th ed.* London: Sage Publications

Hodgen, J. 2018 conference presentation at BCME9 on *Investigating Mathematical Attainment and Progress (IMAP): The Low Attainment in Year 9 Project*

Russell, Susan Jo. "Developing Computational Fluency with Whole Numbers." *Teaching Children Mathematics*, vol. 7, no. 3, 2000, p. 154. *Academic OneFile*, Accessed 24 July 2019

Appendix 1: Collaborative Lesson Research Proposal

Date of lesson: 7th May 2019

School: The Castle, Tiverton

Teacher: Ruth Trundley*

Planning team: Bridget Phillips, Stuart Boon, John Mortimer, Helen Smith, Laura Nethercott, Matt Fullerton, Claire Paton, Amanda Ballard, Ruth Trundley* and Stefanie Burke*

Year group: Y4

The planning team were involved in four sessions across a period of six weeks directly leading up to the live lesson:

- Session 1 focussed on exploring the mathematics (75 minutes)
- Session 2 introduced the particular area of focus, related research and readings and the problem to be considered for the lesson (40 minutes)
- Session 3 involved discussion of the learning from the research and readings and how this influenced thinking about the proposed problem – planning decisions to be made were identified (75 minutes)
- Session 4 planning continued and completed (90 minutes)

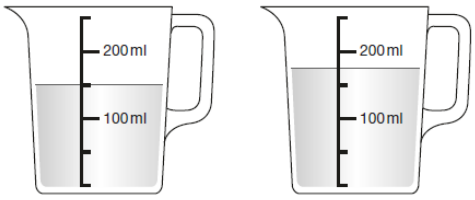
Research theme: ***What are the key elements to understanding number lines as representations of the number system and how does this understanding support children to make decisions when calculating?***

> *Number lines should be used as a central representational tool in teaching number, calculation and multiplicative reasoning across Key Stages 2 and 3…we note the particular value of using manipulatives and representations in principled ways for specific topics, such as the importance of the number line in extending learners' understanding of whole numbers to fractions, decimals and percentages…Bryant (2009) makes clear links to the importance of the number line and the need to recognise that fractions and decimals expand the number system beyond whole numbers.* Hodgen et al 2018

Research shows that pupils in KS3 who struggle with mathematics do not understand number lines as representations of the number system (Hodgen et al 2018). There is no explicit expectation of, or progression in, using number lines as representations of the number system in the Primary National Curriculum; explicit references to number lines are limited to KS1. Consequently, whilst number lines are often used as a tool for solving a calculation in primary classrooms, explicit teaching of the structure of number lines is rare. Without an understanding of number lines, elements of measurement and calculation are undermined, making problem solving difficult.

11 Stefan has **600 millilitres** of water in a bottle.

He pours some of the water into two measuring jugs as shown.



How many millilitres of water are left in Stefan's bottle?

This question from the 2018 national curriculum tests, Key stage 2, Mathematics Paper 3 was answered correctly by 71.6% nationally, fewer than three quarters of the Year 6 children sitting the tests, despite the content being identified as coming from the Year 3 and Year 4 curriculum.

Understanding of the structure of number lines, supporting understanding of how numbers fit together, where numbers are in relation to each other, informs thinking and decision making when calculating. This includes subtraction questions where the numbers are close together. Research shows that children often do not take this into account when the subtraction is presented symbolically:

> *Subtraction problems of the type a – b = ? can be flexibly solved by various strategies, including the indirect addition strategy ("how much do I have to add to b to get at a?")…The results of our second and third study showed that elementary school children seldom used indirect addition on 2-digit subtractions, despite its computational efficiency…*
>
> *(b) All children reported various strategies for solving the two small-difference items from the Variability on Demand Task, but only a minority of them reported IA as an alternative strategy, suggesting that IA was no part of the strategy repertoire of most children. Actually, only 5% of the second graders, 15% of the third graders, and 20% of the fourth graders mentioned the IA strategy at least once as a possible alternative solution strategy.*
>
> *(c) Despite the low overall frequency of the IA strategy in the two tasks and in the three age groups, there tended to be an impact of mathematical achievement level on the frequency with which IA was spontaneously used (in the Spontaneous Strategy Use Task) or mentioned as an alternative solution method (in the Variability on Demand Task): higher achievers tended to solve more small-difference items with the IA strategy than lower achievers.*
>
> Torbeyns et al 2009

…two types of strategies can be distinguished1: (1) direct subtraction strategies, in which the subtrahend is directly subtracted from the minuend (e.g., 75 - 43 = . by '75 - 40 = 35, 35 - 3 = 32'), and (2) subtraction by addition strategies, in which one determines how much needs to be added to the subtrahend to get to the minuend (e.g., 75 - 43 = . by '43 + 30 = 73 and 73 + 2 = 75, so the answer is 30 + 2 = 32'). This subtraction by addition strategy is considered to be particularly efficient on problems with a large subtrahend (S) compared with the difference (D) (e.g., 93 - 88 =, where S and D are 88 and 5, respectively), because the subtraction by addition strategy then requires fewer and/or smaller calculation steps than a direct subtraction strategy (e.g., Torbeyns, De Smedt, Stassens, Ghesquie`re, and Verschaffel, 2009)…

While previous studies have shown that adults use the subtraction by addition strategy frequently, efficiently (i.e., fast and accurately), and flexibly (i.e., mainly, but not exclusively on problems with a relatively large subtrahend) (Peters, De Smedt, Torbeyns, Ghesquie`re, & Verschaffel, 2010a,b; Torbeyns, De Smedt, Peters, Ghesquie`re, & Verschaffel, 2011; Torbeyns, Ghesquie`re, et al., 2009), well-documented evidence on primary school children's self-reported use of the subtraction by addition strategy when mentally solving two-digit subtraction problems is limited.

Peters et al 2012

The children in this study were similar to those in the studies above (see assessment data below); for the calculation 552 – 548 none of the children chose to use 'indirect addition'.

Mathematics sequence: Understanding the structure of the number system and how numbers fit together, in order to be able to measure and calculate efficiently and accurately.

National Curriculum Programmes of Study (and *non-statutory guidance*)

Year 1

Number

- identify and represent numbers using objects and pictorial representations including the number line, and use the language of: equal to, more than, less than (fewer), most, least

Addition and Subtraction

- solve one-step problems that involve addition and subtraction, using concrete objects and pictorial representations, and missing number problems

  Problems should include the terms: …distance between, difference between, more than and less than, so that pupils develop the concept of addition and subtraction and are enabled to use these operations flexibly.

Measurement

- measure and begin to record the following:
  - lengths and heights
  - mass/weight
  - capacity and volume
  - time (hours, minutes, seconds)

  Pupils begin to use measuring tools such as a ruler, weighing scales and containers.

Year 2

Number

- identify, represent and estimate numbers using different representations, including the number line

  As they become more confident with numbers up to 100, pupils are introduced to larger numbers to develop further their recognition of patterns within the number system and represent them in different ways, including spatial representations.

Addition and Subtraction

- solve problems with addition and subtraction:
  - using concrete objects and pictorial representations, including those involving numbers, quantities and measures
  - applying their increasing knowledge of mental and written methods

Measurement

- choose and use appropriate standard units to estimate and measure length/height in any direction (m/cm); mass (kg/g); temperature (°C); capacity (litres/ml) to the nearest appropriate unit, using rulers, scales, thermometers and measuring vessels

Year 3

Number

- identify, represent and estimate numbers using different representations

  Using a variety of representations, including those related to measure, pupils continue to count in ones, tens and hundreds

Addition and Subtraction

- solve problems, including missing number problems, using number facts, place value, and more complex addition and subtraction.
  Pupils practise solving varied addition and subtraction questions.

Measurement

- measure, compare, add and subtract: lengths (m/cm/mm); mass (kg/g); volume/capacity (l/ml)
  Pupils continue to measure using the appropriate tools and units, progressing to using a wider range of measures

Year 4

Number

- identify, represent and estimate numbers using different representations

Addition and Subtraction

- **solve addition and subtraction two-step problems in contexts, deciding which operations and methods to use and why.**
  Pupils continue to practise…mental methods… with increasingly large numbers to aid fluency

Measurement

- estimate, compare and calculate different measures

Year 5

Number

- interpret negative numbers in context, count forwards and backwards with positive and negative whole numbers, including through zero
  They continue to use number in context, including measurement. Pupils extend and apply their understanding of the number system to the decimal numbers and fractions that they have met so far.

Addition and Subtraction

- solve addition and subtraction multi-step problems in contexts, deciding which operations and methods to use and why.
  They practise mental calculations with increasingly large numbers to aid fluency

Year 6

Number

- use negative numbers in context, and calculate intervals across zero

Addition and Subtraction

- solve addition and subtraction multi-step problems in contexts, deciding which operations and methods to use and why.
  They undertake mental calculations with increasingly large numbers and more complex calculations.

About the mathematics in this sequence

The sequence has a key understanding running through it:

Understanding that structured number lines have regular spacing and that the spaces relate to a set unit (understanding number lines as measure).

The structured number line is a 'measurement model'. By this we mean that numbers are representations of lengths, rather than simply points on a line that they are labelling. The proximity or distance of an 'unknown' number to a known number is important because this enables a student to determine the unknown. The lengths or proportion of the line between the numbers and marks provide clues as to the value of a missing number. Bobis 2007

About the students

Assessment Question	Correct	Incorrect
 The ribbon is blue and red, how long is the red part of the ribbon?	27%	73%
8 + 2	100%	0%
3 + 3	87%	13%
552 – 548 No one looked at the difference. 73% used written method - of these 36% correct. 20% partitioned one or both numbers – 100% correct	47%	53%
837 + 6 40% used written method – of these 83% correct. 33% partitioned 6 – 100% correct	67%	33%

Plan for the sequence

- Taking steps of equal size:
    - Marking steps on lines
    - Exploring horizontal and vertical lines
    - Exploring steps of 1m, 100cm and giant steps of 1000cm

- Moving from one mark to another
    - Understanding what is the same and what is different about numbers between two marks
    - Understanding what has been added/subtracted when moving between two marks linked to the number of steps taken or the number of spaces moved across.
- Identifying numbers on number lines with **ten** steps of equal size:
    - Number lines from 0 to 100, 0 to 1000
    - Making links to different parts of the number system:
        - Ten steps from 0 to 100 linked to 100 to 200 etc.
        - Ten steps from 0 to 1000 linked to 1000 to 2000 etc.
- Identifying numbers on number lines with different numbers of steps of equal size:
    - Number line 0 to 10 with ten steps, five steps and two steps, then 10 to 20, 20 to 30 etc.
    - Number line 0 to 100 with ten steps, five steps, four steps, two steps, then 100 to 200, 200 to 300 etc.
    - Number line 0 to 1000 with ten steps, five steps, four steps, two steps, then 1000 to 2000, 2000 to 3000 etc.
- Exploring and using mid-points
    - Link to taking two steps – halfway
    - Lines with mid-points and quarter-points marked – how to identify the latter through using understanding of mid-points
    - Using understanding of mid-points with lines where the spaces are not all equal
- Locating points between marks
    - Identifying the mid-point to help locate numbers
    - Identifying where to place numbers in relation to a boundary
    - Identifying numbers marked between boundaries
- Linking position of numbers to decision making when adding and subtracting including asking how close together numbers are linked to subtraction

Today's lesson: final part of the sequence

> *…if counting-up is not explicitly taught as a procedure which can yield answers to subtraction operations, children will not have a choice in what procedure to use until such times as their understanding of the relationship between numbers and their constituent parts is sufficiently advanced to let them see for themselves that subtraction is the inverse operation of addition. The possibly lengthy time taken by children to discover for themselves (as against the possibly shorter time which might be needed for systematic teaching) that counting-up is an appropriate way in which to carry out subtraction operations may well account for the fairly common finding that subtraction is more difficult than addition.*
>
> **Maclellan (1995)**

During the exploration of the mathematics in this teaching sequence, the planning team explored problems involving subtraction where the numbers are close together. When these were solved by teachers using addition, they were also represented symbolically as additions; the calculation identified was not the calculation that matched the problem but the calculation that matched the method chosen for solving the problem. From this it became clear that, to quote Maclellan, children are not being explicitly taught counting-up "as a procedure which can yield answers to subtraction operations."

The planning team all read chapter 7 from *Mathematics Explained for Primary Teachers 5th Edition* (Haylock and Manning 2014) in order to explore further the complexity of understanding subtraction. Of particular interest were the different structures of subtraction described in the book as:

- *the partitioning structure;*
- *the reduction structure;*
- *the comparison structure; and*
- *the inverse-of-addition structure.* p. 91

The planning team identified that the authors' description of what is involved in making sense of these structures resonated with what they had discussed when exploring the maths themselves and the research around children **not** choosing to count on to solve subtractions, cited above:

> *There is a daunting range of situations in which we have to learn to recognize that the appropriate operation is subtraction…Each of these has its own characteristic language patterns, all of which have to be connected in the learner's mind with subtraction.* p. 91

During the discussion, the difficulty of connecting the comparison structure with the subtraction symbol, as highlighted during the exploration of the maths, was further explored and the teachers found the following suggestion from Haylock and Manning to be useful:

It helps us to connect these mathematical structures with the operation of subtraction if we ask ourselves the question: what is the calculation I would enter on a calculator in order to solve this problem? In each case the answer will involve using the subtraction key. p. 92

The planning team looked at the calculation 82 – 77 and considered the understanding and knowledge needed to solve this efficiently using a counting-up method. This included:

- Knowing that 82 and 77 are close together
- Know that you can solve the problem by finding the difference
- Knowing that both 82 and 77 are close to 80, which lies between them
- Knowing that it is easy to step from 77 to 80 (using known fact 7 + 3 = 10)
- Knowing that it is easy to step from 80 to 82 (using place value knowledge)
- Keeping track of the different steps undertaken
- Knowing that the two steps taken together show the gap between 77 and 82
- Knowing that 5 = 3 + 2

Much of this understanding and knowledge develops from understanding the structure of the number system as represented by a number line.

In *Teaching for Mastery: Questions, tasks and activities to support assessment Y4* (Askew et al 2015), the big ideas for addition and subtraction include:

Looking at the numbers in a calculation and their relationship to each other can help make calculating easier. For example, 3012 – 2996. Noticing that the numbers are close to each other might mean this is more easily calculated by thinking about subtraction as difference. p. 12

About the lesson

Goal: Pupils can think about the meaning of comparing two prices and connect this to subtraction, using their understanding of number lines and calculation to explain and represent their thinking in the context of solving two-step problems

The planning team discussed at length the language, numbers and context for the problem.

The vocabulary in the problem was changed from deal to offer. This was decided because deal has many different meanings in different contexts (deal out cards, I'll deal with you later, no deal…) and may be a word the children are less familiar with in the context of shopping. It was planned for the word 'offer' to be introduced and explored in the previous lesson and the children given homework to see if they could find any special offers in a supermarket.

The planning team considered the importance of language associated with 'special offers'; this language helps to make sense of what you expect from a special offer and therefore a reason for the mathematics in the problem. The tendency to focus on 'more' rather than 'less' (explored in the readings) may influence the way the children talk about what they expect from a special offer, i.e. that they will get more rather than it will cost less. The team identified that these are the two different types of offers in supermarkets: more for the same money (for example, buy one get one free, three for the price of two…) and reduced price (for example 25% off, two for £2). As the focus of the session is a special offer that saves money, it was decided that only the language associated with saving money would be captured. The word 'less' is used in three of the anticipated phrases but in different ways; less expensive was identified as a potentially problematic phrase because the two words seem to indicate opposite things about the price.

The word 'compare' was considered important in the context of the mathematics focus of the lesson and needed time given in the lesson to allow the children to make sense of the word when used in relation to the problem being explored.

The context of bottles of orange juice was discussed. Squash was considered as an alternative, as being a drink the children might be more likely to have at home, but the requirement of the problem, related to buying juice every week, seemed less realistic for squash. The problem was changed to buying two bottle every week rather than one bottle a week for two weeks to reduce the complexity of the situation without reducing the complexity of the mathematics in the problem.

The planning team chose £1.30 for the price of one bottle, anticipating this would be straight-forward for the children to double; as the focus on the session is connecting comparing prices with writing a subtraction, this was considered important. The problem needed to be two-steps in order to fit with the expectations of the Y4 curriculum, and the team was interested to provide an opportunity for children to apply their understanding of comparing by adding on in a two-step problem and then make the connection to subtraction. The special offer was originally suggested as £2.25; £2.35, £2.45, £2.39

and £2.49 were all considered. It was agreed that a price closer to £2.60 than £2.25 would be a better choice. There was discussion about whether the number should end in five or nine. Fives have been linked to mid-points during this sequence and the Y4 children are fluent at counting in fives; it was decided that this might take attention to the counting rather than thinking about the gap and what it means in the context of the problem. £2.39 was chosen as demanding slightly more thinking and because it was anticipated that some children might partition into £2.30 and 9p and add on 30p and 1p to get an answer of 31p.

It was decided to start from a numberless problem, so that the children could make sense of the context and start thinking about it mathematically before they started to calculate. The team anticipated a variety of methods and solutions for the problem but decided that these should not all to be shared. The focus is on the correct solution, how it can be efficiently found and the accompanying calculations. Whilst the children will be encouraged to share their solutions and all will be captured, the focus will then move to exploring the correct solution. The children will have the opportunity to consider their first answer and reflect on it later in the lesson.

It was decided that the problem would be modelled with two bottles of orange juice and labels similar to those found in supermarkets as well as the problem displayed in words and that the children should have a picture of the priced bottles alongside the written problem, in order to support access to the problem.

For the first calculation the decision was made to record it as an addition (£1.30 + £1.30), partly because it is anticipated that many of the children will record it like this, even if they use language such as ‘double’, and partly because the order of writing multiplications can be problematic and this is not the focus of the lesson. However, if children do record it as a multiplication the equivalence between the two recordings will be noticed; this formed part of the previous lesson.

There are two elements to the main part of the lesson, establishing the calculation that accompanies the comparing of the prices, i.e. how do you symbolically record comparing prices, and connecting noticing that two numbers are close together in a subtraction leads to a decision to solve it by adding on. To support this latter element, the decision was made to have pre-prepared number lines with the prices on to model and for the children to use themselves at the point this is being discussed, not when they are deciding how to solve the problem. Consideration was given to different number lines and the decision made based on the number line being used to explore thinking that has happened and understanding being shared rather than ‘doing’ the calculation.

Flow of the Lesson

What will the teacher **say and do**? What will be recorded and how (plan for the board)?	Anticipating: What do we think the children **might** say and do?
Introduction • What sort of special offers have you seen in supermarkets? Talk to your partner.	Easter eggs offers e.g. 3 for £10 Save 25% Buy one get one free Buy one get one half price Three for price of two Two for two pounds Sweets in Tesco Crisps
• What makes it a 'special offer'? Why do people like special offers? Use the sentence starter 'I think …' and talk to your partner. Include these words on the board if suggested: • Save money • Cheaper • Costs less • Spend less • Less expensive • Lower price	Cost now, cost before, saving of … Save money Some deals you get more Some are cheaper It costs less Spend less More for your money Free items, free stuff, extra free

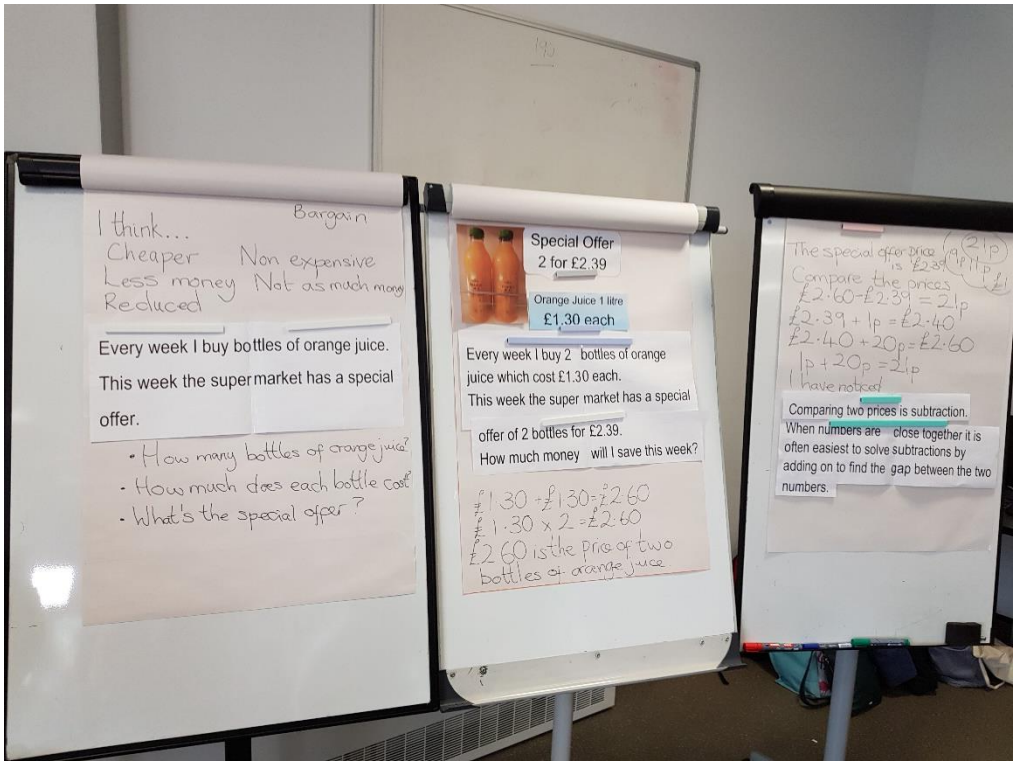
What will the teacher **say and do**? What will be recorded and how (plan for the board)?	Anticipating: What do we think the children **might** say and do?
Presenting the problem • Every week I buy bottles of orange juice. • This week the supermarket has a special offer: Is this a maths problem? What questions do you have that would make it a maths problem? Talk to your partner. Monitor, select and sequence the three questions needed for the lesson. Prime children to share these. If don't hear them – pretend to have heard them. ○ How many bottles are you buying? ○ How much does one bottle cost? ○ What's the offer? • Every week I buy 2 bottles of orange juice which cost £1.30 each. • This week the supermarket has a special offer of 2 bottles for £2.39. • How much money will I save this week? On board and sheet on table for all children – pictures and words	Why does some juice come in bottles and some in cartons? How much is in a bottle? How much does one bottle cost? Is it Tropicana? Is the orange juice with bits/without bits What is the offer? Why is it an offer? How many bottles are you buying?
Problem solving by the students • Have a go at solving this problem on your own – write or draw anything you want to help you.	**Step 1** £1.30 + £1.30 or £1.30 x 2 solved by: • mentally/jotting the addition of £1s and the 30ps • column addition • counting (fives or tens) Possible errors with this stage… £1.60, £2.30, £2.06

What will the teacher **say and do**? What will be recorded and how (plan for the board)?	Anticipating: What do we think the children **might** say and do?
• Think about the calculations you used. Why did you do these? Write or draw something to show your thinking about how to solve the calculations.	**Step 2** £2.60 – £2.39 solved by: • counting up from 2.39 to 2.60 • column subtraction • adjustment e.g. 60 subtract 40 plus 1 Some children might not choose to do two steps or may do the first step correctly but get the next one wrong e.g.: • 1.30 + 2.39 • 2.39 - 1.30 • 2.60 + 2.39 Some children might draw a number line for one/more steps Some children might draw sticks and dots for one/more steps
Comparing and discussing (neriage) • How much money do you think I would save? Record all solutions on the board. • We are going to see if we can agree that I would save 21p – don't worry if you got a different answer, if you agree it is 21p when we have discussed it I will give you time to think about why you got a different answer at first. • I saw lots of people do two calculations. What was the first calculation you did? Have a look. Choose someone who doubled £1.30 to explain and write £1.30 + £1.30 = £2.60 on the board. Do you agree? Because…	Solutions children could suggest: 9p, 39p, £3.69, £4.99, £1.09, 21p, 31p If children have also written £1.30 x 2 acknowledge this here and ask how the calculations are the same.

What will the teacher **say and do**? What will be recorded and how (plan for the board)?	Anticipating: What do we think the children **might** say and do?
• Ask: What is £2.60? • £2.60 is the price for two bottles of orange juice. The special offer price is £2.39.	
• What do I need to do now? Talk to your partner. • Listen for what the children say.	Finding how much more/less Finding the gap Finding the difference
• I have heard people say I need to **compare** the prices. What does this mean? What are we trying to find out? Write 'Compare the prices' on the board. • What calculation shows we are comparing the prices? Write = 21p and say, we know the answer is 21p, that we would save 21p, so what is the calculation that tells us this? Write it down. Share with your partner – do you agree?	Finding out how much cheaper/more expensive
• Why is this the right calculation for the problem? Talk to your partner. • Prime someone to explain – choose someone else to repeat or repeat together.	£2.60 - £2.39 = 21p £2.39 + 21p = £2.60 £2.60 + £2.39 = 21p Look for the correct calculation
• Let's think about how to solve this problem. Ask a child who got 21p by adding on to explain what they noticed (that the numbers are close together). Do you agree? Are the numbers close together?	

What will the teacher **say and do**? What will be recorded and how (plan for the board)?	Anticipating: What do we think the children **might** say and do?
Show on a number line marked with £1 steps from 0 to £3 and labels: £2.39, £2.60. Discuss in pairs with a number line. • Ask another child who added on (in two steps) to explain how they found how much could be saved. Model this explanation on a number line showing that the numbers are close together and record the matching calculations. Children move their finger on their number line to match this. • Do you agree? Use your number line to explain why you agree/disagree. • Now look at how you worked out the answer – what have you noticed now we have worked on the solution together? Write I have noticed…if 21p is different to your first answer, can you see why it is different; draw an arrow to show where you did something different. • Ask children to complete the sentence: The special offer… - model this first	Children using the language shared earlier
Summing up • Comparing two prices is subtraction. When numbers are close together it is often easiest to solve subtraction by adding on to find the gap between the two numbers. I am comparing these pairs of prices. Which ones would you solve by adding on? Why? • 96p – 7p = • 96p – 92p = • 96p – 87p =	

Lesson

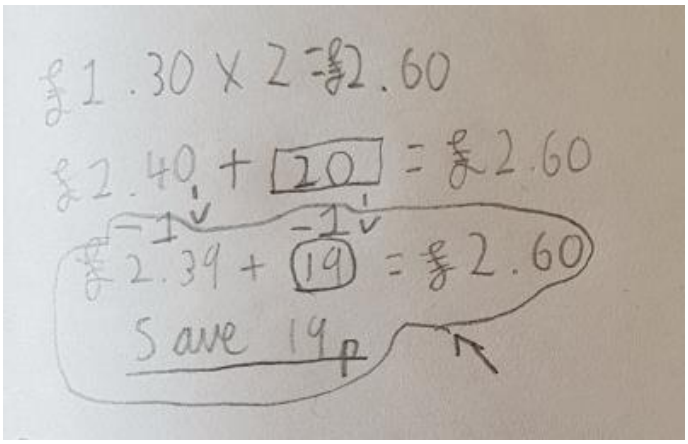

I think...
Bargain
Cheaper
Non expensive
Less money
Not as much money
Reduced
Every week I buy bottles of orange juice.
This week the super market has a special offer.
- How many bottles of orange juice?
- How much does each bottle cost?
- What's the special offer?
Special Offer
2 for £2.39
Orange Juice 1 litre
£1.30 each
Every week I buy 2 bottles of orange juice which cost £1.30 each.
This week the super market has a special offer of 2 bottles for £2.39.
How much money will I save this week?
£1.30 + £1.30 = £2.60
£1.30 x 2 = £2.60
£2.60 is the price of two bottles of orange juice
The special offer price is £2.39
Compare the prices
£2.60 - £2.39 = 21p
£2.39 + 1p = £2.40
£2.40 + 20p = £2.60
1p + 20p = 21p
I have noticed
Comparing two prices is subtraction.
When numbers are close together it is often easiest to solve subtractions by adding on to find the gap between the two numbers.

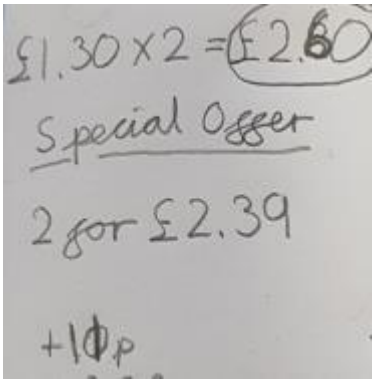
Post-lesson Discussion

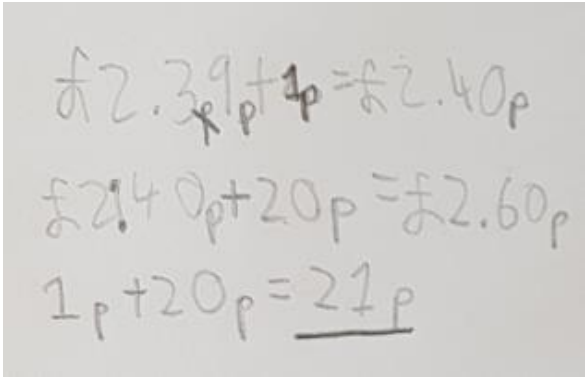
There was an opportunity to look at the children's work and share observations in groups before the formal post-lesson discussion began:

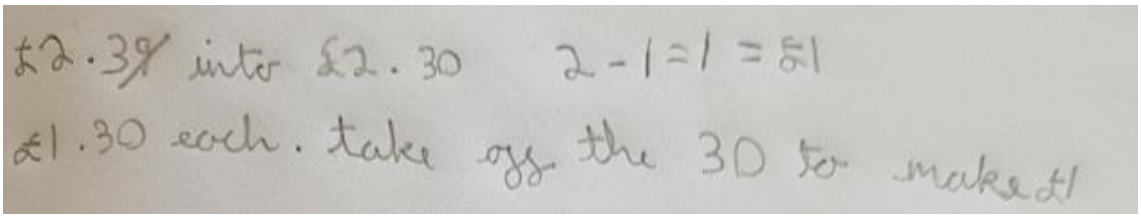

Key areas identified in the post-lesson discussion:

- Anticipation
    - The planning team anticipated various errors and potential misunderstandings but many of these were not observed; many of the children successfully solved the problem. The number of different solutions offered was limited to: 11p, 19p, 21p and £1.

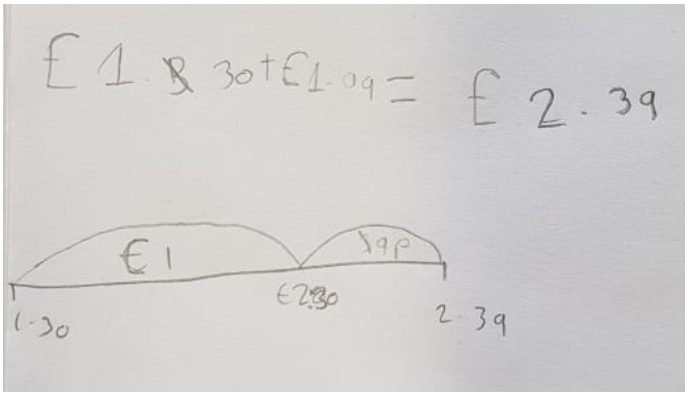




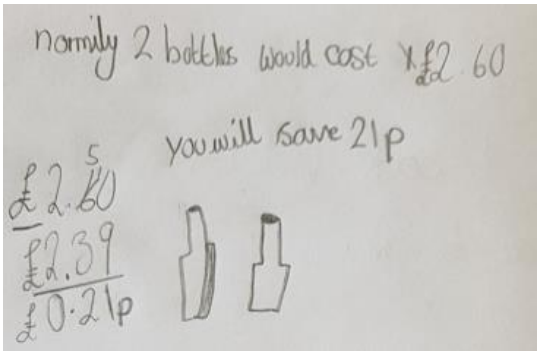




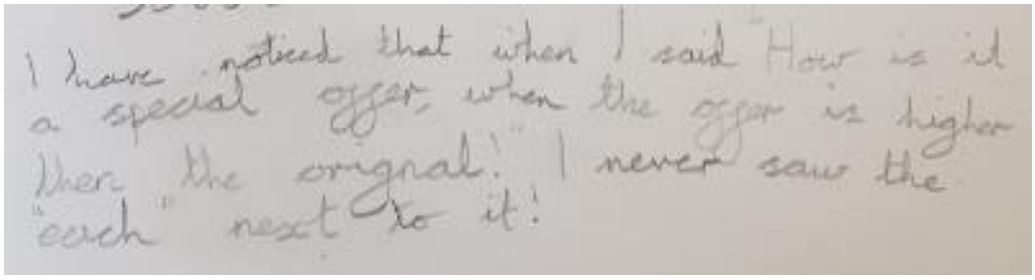
Not all methods were anticipated:



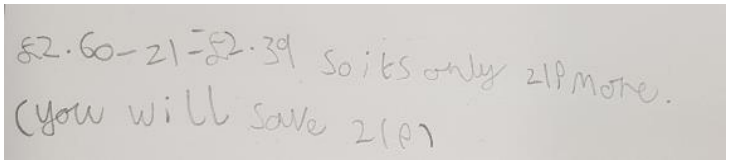
Only a few children used a standard written method.

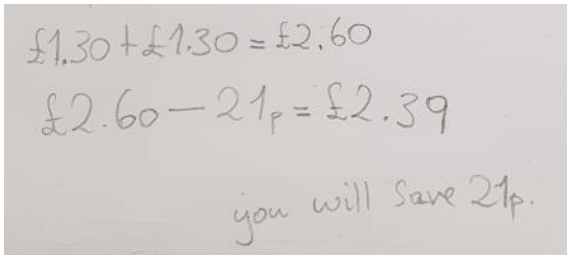


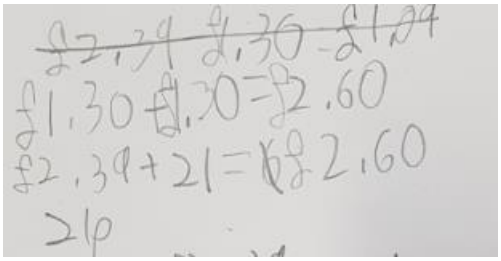
The team did anticipate that some children might not think it was a special offer and when the problem was first presented one child said that they didn't understand because the special offer cost more than the normal price. At the end of the lesson she reflected on this:



- The anticipated difficult part of the lesson did prove to be difficult. Whilst many of the children solved the problem, they did not connect it to a subtraction focussed on comparison. Most recorded either an addition or a subtraction taking away the saving.

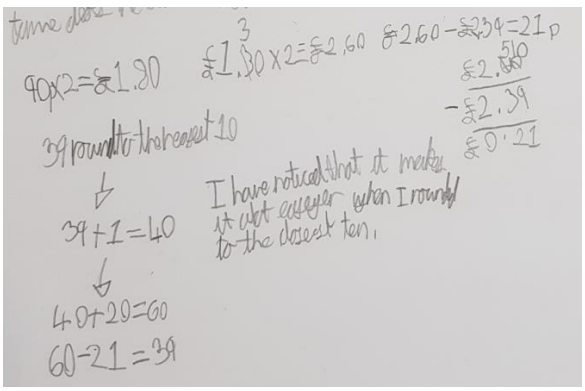


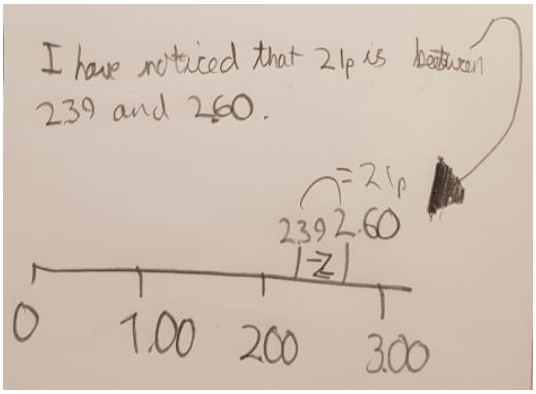


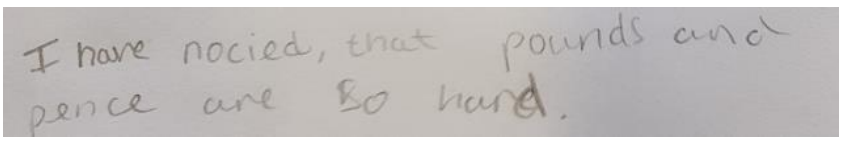


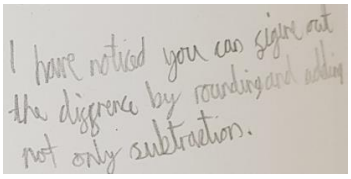
- Language that is incidental is important.
  - 'I have noticed…'

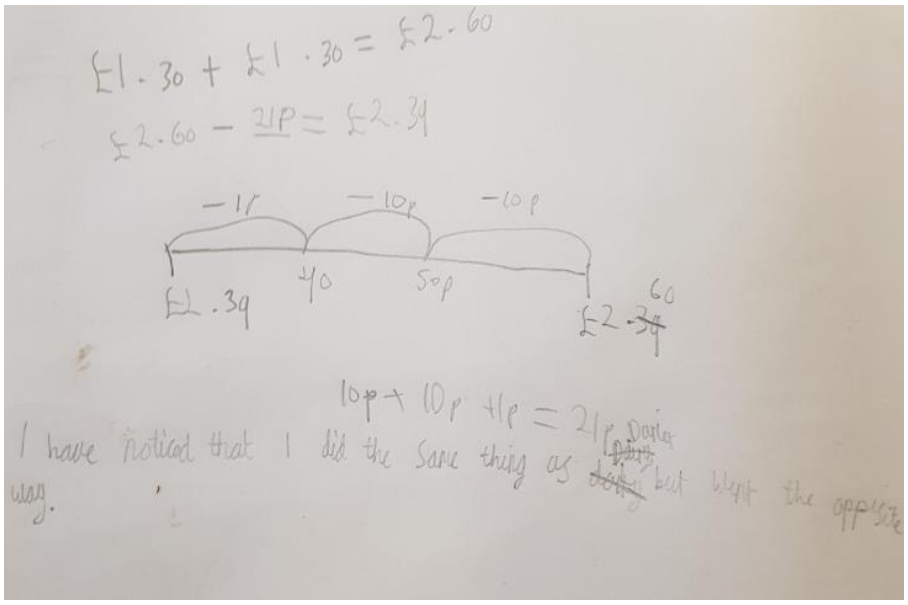
    The children were not necessarily clear on what it means to 'notice'. The planning team were asked about what they were expecting. The intention of this opportunity to reflect at the end of the lesson was for the children to go back and look at their thinking about how to solve the problem and their solution compared with the thinking and approaches shared in the lesson to see if they now thought differently. Some children struggled and their noticing was focussed on less consequential aspects of the lesson whilst others did reflect on the mathematics that was the focus of the lesson. It was suggested that for children to be able to write what they have noticed, they need to know that they have noticed something. This can be prompted through questioning and needs to be modelled.



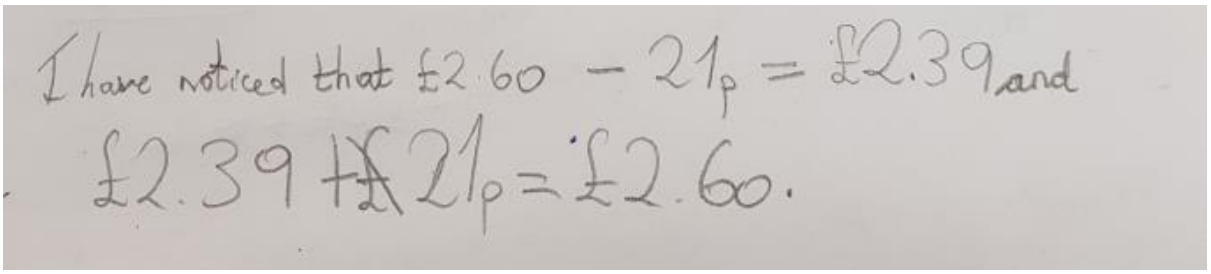






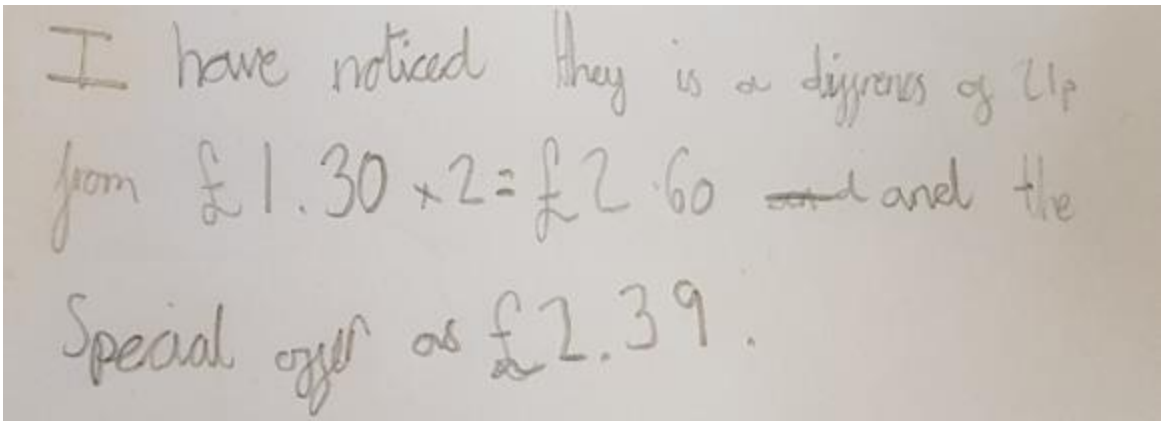


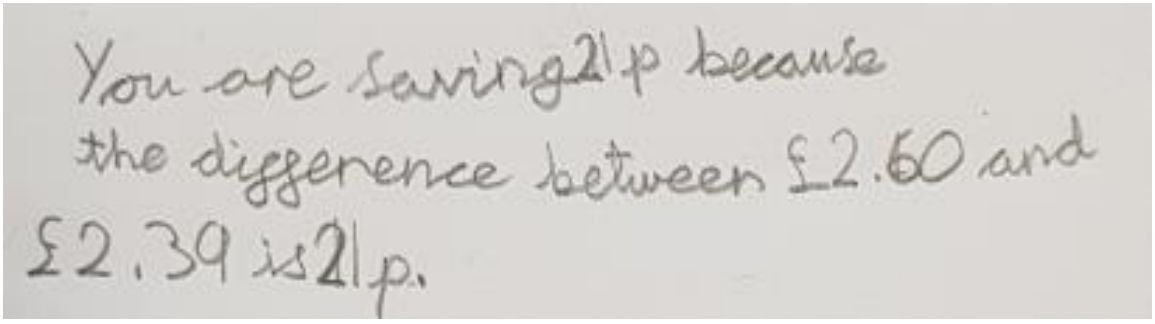
The use of this sentence starter was also an opportunity to reflect on connecting the problem with recording it symbolically as a comparison subtraction. The planning team did not plan for this explicitly; there was greater potential with using this sentence starter than was fully realised.



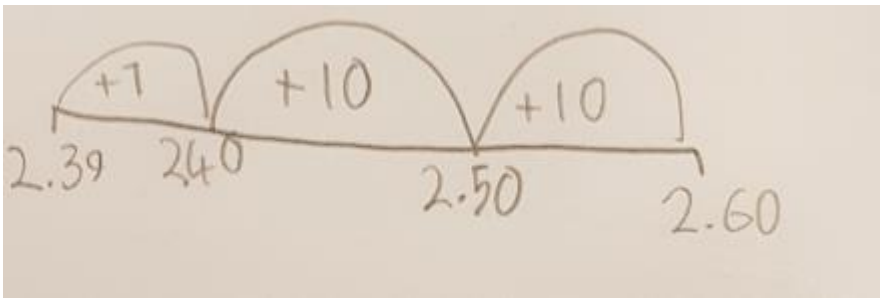
- In general it was some of the incidental language that proved to be most challenging; one observation was that the planning team had given a lot of attention in their thinking to the mathematical language related to the context but not explored, in the same depth, the language of explaining, reasoning and critiquing. It may be of great benefit to learners if teachers attend to progression in this language as well as mathematical vocabulary.
- The planning team had written ‘compare’ into the plan and it was planned for the teacher to listen for this being used. However, in the lesson no-one was heard to use ‘compare’ but someone offered ‘difference’. Instead of following this, and establishing the understanding of the context from this word, the teacher stuck with the word from the plan and introduced ‘compare’. This meant that the focus on meaning was undermined; the language that had emerged from the children was not explored. A more thorough exploration of what it means to find the difference in the context of the problem would have led to an understanding that it involves comparing prices in order to find out how

much more/less the orange juice cost. Some children did use 'difference' in their writing about the problem:

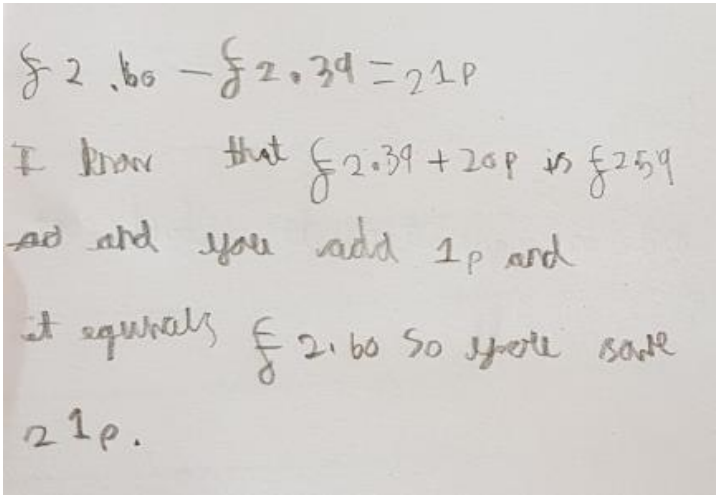




- Showing thinking is challenging
    - Linked to noticing, the children were not entirely sure how to 'show' their thinking. This was another example of language used where the intention was not fully understood by the children.
    - Modelling with a number line - there was discussion about whether the children should have been directed to show their thinking 'on a number line' or asked to use a number line. The planning team made a deliberate decision not to ask the children to use a number line; the intention of the lesson was to see if the children applied their understanding to the problem and connected this to subtraction. When the number line was introduced there was an opportunity for the teacher to model the connection between the problem, the symbols and the method. Unfortunately, the number line prepared for use by the teacher for the lesson was not big enough for all the children to see; although they had a number line in front of them for looking at in the latter part of the lesson, modelling for the whole class how one child had added 1p then 20p on the number line did not take place. There were examples of children who chose to show their thinking using a number line; a visualiser would have allowed these to be shared with the rest of the class.



- When children didn't articulate their thinking, it led to observers making assumptions about their understanding and involvement in the lesson. The opportunity to reflect and write at the end can reveal thinking that has taken place. One child was observed to be seemingly little involved, choosing not to talk to her partner, but at the end her writing revealed her thinking:



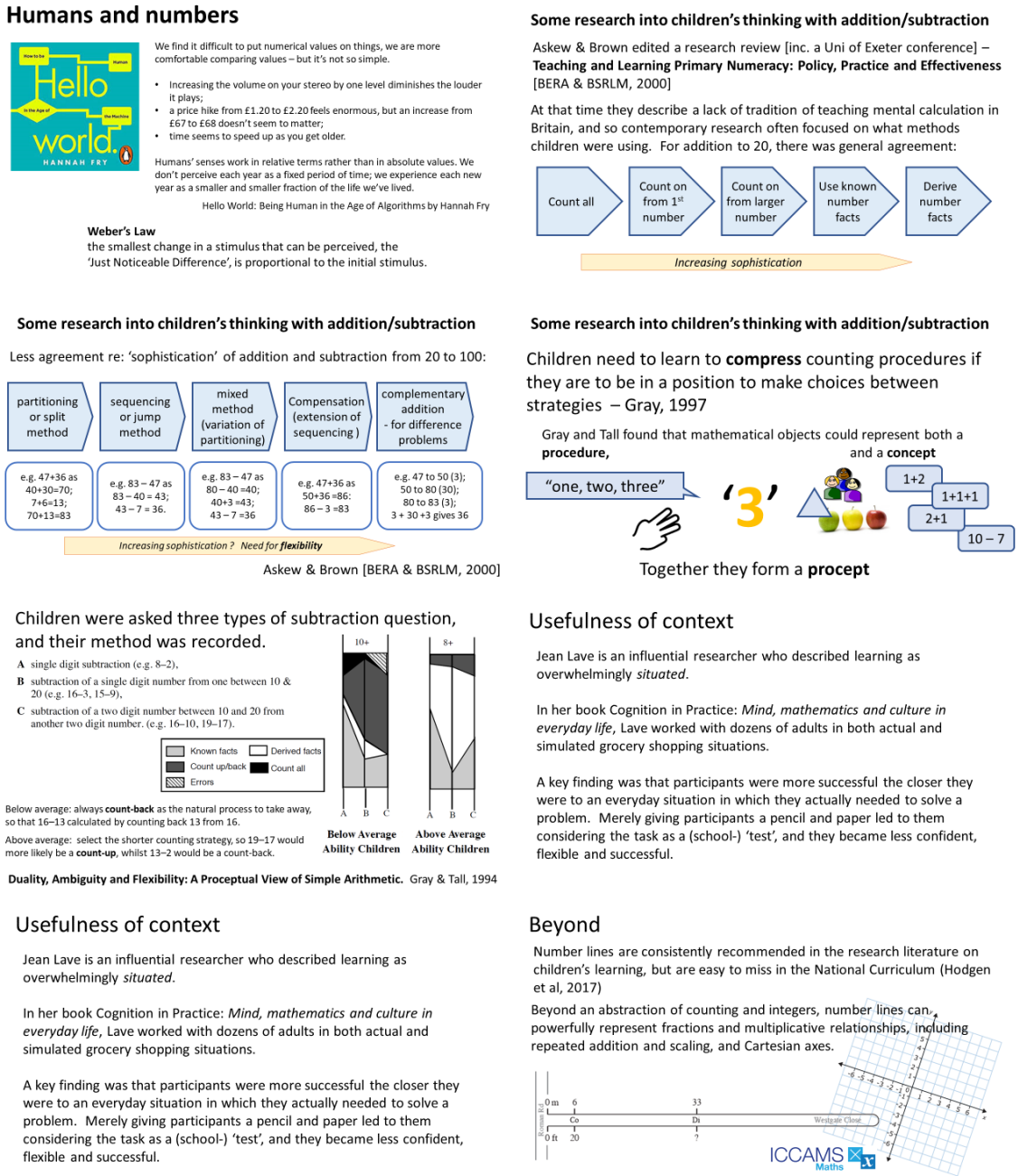
- Where children did not show their thinking in their books it did not mean they didn't do any thinking. As with observing, a lot of assumptions can be made when looking at books. Listening to the teacher talking to children is important in order to know what they are thinking. One child said that pence and pounds can be the same and that 2p and £2 was the same. When asked which she would rather have she said '2p'. Eventually she explained that 2p was cheaper than £2 and she was thinking she would rather have this as an offer; initially her language did not match that of the teacher's and meaning had to be negotiated.

Koshi – summary

The planning team were congratulated on a 'very well thought through research proposal' and attention was drawn to a key moment in the lesson, which had not been planned for. This was when the teacher asked the children to identify where the 21 was on the number line. The Koshi placed the lesson in the context of research into:

- Thinking about addition and subtraction
- Number lines
- Usefulness of context

Slides used to link to this research are shown below.



Reflections on the CLR cycle

All participants completed a reflective feedback survey after the lesson, discussion and Koshi.

Reflections from the planning team included:

- It was incredibly valuable because it really highlighted how important each part of a lesson is. For example, the language choices, the numbers chosen and the impact that the board work can make. Working collaboratively with teachers across the age ranges helped to make a thorough lesson plan that was very focused.
- From the perspective of being in the planning team, which was really enjoyable as well as a great PD opportunity, I think, as discussed we needed more time to plan. Another session would have really helped. The build-up and the lesson itself have highlighted for me the importance of language and children's understanding of this. The discussions in planning and after the lesson reinforced this. Secondly, and particularly watching and listening to the children today, it highlighted the importance of exploring all the structures of subtraction from an early age. Finally, the number line appears to me a really valuable asset to children in two ways. Firstly, for them to be able to visualise numbers and a number line in their head to be aware of distance and proximity between numbers, both whole numbers and decimals/fractions. Secondly, to use as an aid / strategy / way of demonstrating their understanding when solving problems. The children today didn't have to draw a number line to show they understood the linear proximity of the numbers involved. By finding the gap between them, many were demonstrating their awareness of the proximity of the numbers - the number line may have been in their head and may have been the reason they weren't 'taking away' £2.39 from £2.60.
- It has been a great experience to be involved in the whole process and to be able to see it through to the end. It was great to see the live lesson and to see our planning in action. The main outcome for me was the importance of language. Number lines still need to be explored further and continue to be used in daily maths when appropriate to consolidate the benefits of using them
- The benefits of using research to inform professional practice were apparent in this process, e.g. research related to different addition and subtraction structures and research reflecting how children find it challenging to use indirect addition when finding the difference. It would be good for schools to have access to high quality research. I have also taken from it the need to discuss with pupils whether it's more efficient to count on or back depending on how close the numbers are. This is already having a positive impact in Y2.
- Being involved in the planning was really interesting. As upper KS2, it was fascinating & useful to hear insights/perspective from KS1 colleagues. Being involved in planning the lesson also helped me to follow the lesson with good understanding. I wonder if it seemed so clear to colleagues who weren't involved and didn't necessarily understand the decisions we had made.
- It was a very well thought out lesson and being part of the planning team was beneficial. Observing how the things we planned impacted on the lesson was very beneficial. I feel as though more time would have been needed for the planning (but we already know this) to allow us to finalize some minimal details within the lesson.

- It was a very informative experience to think in detail about every part of a lesson which I do on a daily basis. The impact of subconscious decisions is huge and has allowed me to reflect on the impact it has on my own practice. It was great to see through the whole process and see how the live lesson reflected what we had planned.
- The process of planning successful maths lessons is very complex and requires careful thought about many aspects. However, the lesson itself will also sometimes be quite fluid and may evolve in a different way to the initial plan as the children's understanding/misconceptions are revealed. Discussion is a very important part of successful planning and it is key to focus on the language being modelled and the methods that we hope the children will be able to use over time. Those who have a good understanding of number need to be challenged to perfect their explanations regarding their methods and understanding of how they have reached their conclusions.

Reflections from other participants included:

- Engaging process particularly from a non-maths specialist perspective. Pre-reading information was extremely useful for context and initial development of understanding of focus of lesson. Being able to listen to the children's discussions and review their learning supported the connection between theory, input and learning
- Interested in how teachers' use of language impacts on children's ability to access the information. It was great to see the amount of time spent considering this and how we consider this further in all areas. Interested in learning more about the use of number lines in early years to improve this area of maths learning i.e. what are the building blocks?
- I was really interested to see the amount of time and attention placed by the research team on the minutiae of the lesson, in particular the language used. Talking to some of the research team they were clear that the importance of the language merited the amount of time and effort spent on it. Being part of the lesson though has made me think hard about where the most productive focus may lie in this kind of professional development experience. I will certainly be adopting some of the attention to detail used by the research team, drawing on the use of the technique of distinguishing explicitly between expected and actual outcomes. I can see how powerful that could be.
- The choice of the task/ research was great, thank you for choosing it. In theory these comparing questions should be straight forward - we are all bombarded with offers everywhere, TV, Shops, Internet, so children/students can see their real life application... yet these are the questions that many students answer incorrectly or are unable to justify. There is clearly a gap in the understanding of how number lines are used / not used. The Lesson Study also highlighted to me the importance of keeping a very focused goal in the lesson. I always get surprised at how quickly the time goes in the Lesson Study, hence I think it's important to have a very clear 'narrow' target. Finally a lot of work had gone in the lesson we saw today, so thank you!!! I have definitely learned a lot and will go over your research and planning again.

- It would be great to have film of the planning sessions. Then maybe significant bits of this could be included in any pre-reading / watching along with the research proposal. I was left with some rich questions that I want to work on in my own thinking.
    - When we ask pupils what did they notice or to show their thinking, etc. there may be work we need to do in helping them know what that means. They may notice a number of things - how do they choose which are the pertinent things to notice. When I show my thinking, what aspects of my thinking is it useful to show in the context of the lesson. This is all tied up for me about learning objectives - how is this communicated to pupils in ways that make it situated and real for the pupils?
    - What is the role of context? Does it support or hinder? Contexts help make the learning meaningful and therefore memorable but also root the knowing in particulars. Given a key aspect of mathematics learning is to generalise how can a context be chosen in such a way that when the context is removed the essence of the (abstract) concept remains (as in RME)?
- I really enjoyed this experience and wish that I could have been a part of the whole experience! It was great to watch this lesson as part of sequence. As a school we are also trialling out this number line research and for me I enjoyed watching this lesson as a part of that. I also felt that the discussion afterwards with the links to academia was invaluable and something I would like to continue within my own PD at school.
- It was a great opportunity to see the CLR in practice and fascinating to see the children's responses to the stimuli, instruction and questioning. It became clear from my observations that whilst most children had successfully used a variety of informal and formal subtraction strategies to find the difference between two prices and compare the offer, the link made with their understanding of number lines and calculation was not as strong as expected and few drew on previous work. For me on reflection of my observations of the children's responses and work examples I saw, it reaffirms my view that when lesson designs are being crafted, it is vital to consider what it is that we want children to specifically attend to … and then to carefully choose a model and representation that will support and draw focus to this. Close attention was paid to the use of vocabulary, and probing questions gave many opportunities for the children to think and articulate their understanding. However, from what I observed many struggled to explain their thinking and the processes they went through in both words and/ or pictures. I think that this is something that again children need to have carefully scaffolded and repeatedly so over time, for metacognition to become second nature.
- Focusing our observations on just one group led us noticing more about the children's responses and learning behaviours which led to a richer discussion. So much of a child's success in maths is dependent on them understanding a range of contextual and mathematical language and this has to be so carefully planned for as we make so many assumptions. I hadn't thought about the word 'noticing' in such a way before this experience and how we need to really explore it with children

as they can be closed in their thinking if playing 'I wonder what it is my teacher wants me to notice? I think the decision that the children would work individually first was a really good one as not sure had the children been given the opportunity to calculate and explain in pairs there would have been as many different written responses.

- A brilliantly organised and planned event, highly professional, and a great learning experience for participants. I have a few observations to throw into the mix.
    - It was great to see the flow of the lesson unfolding on the flip charts... the critical importance of what children see in a lesson is so often under-planned, so it was great to see it unfold on the charts.. such a shame we don't have those wonderful expansive boards they have in Japan.
    - The research proposal was brilliant, it gave such helpful background to your thinking, and how you arrived at this lesson.
    - Also really liked the clarity of the unpacking of the steps…both for the sequence of the development in number lines, and the detail of the 82-77 example.
    - There seemed to be several threads that emerged: comparing and 'take away' models.. each of which generated a very different image on a number line.. this did not emerge in the lesson although some children (I think) subtracted by partitioning the second number, which is essentially the take away model... but most counted up or down between the two numbers; another thread was when do you decide when to 'find the difference' and when do you not.. maybe the visual of the take away vs difference models as seen on the number line might support this decision making in a future lessons; another thread seemed to be about 'showing' your thinking on a number line (whether or not you did the calculation in your head).. which is a sort of 'joining the dots' thing that children often find difficult.. ie how does this bit of learning link with that bit.
    - The discussion about the importance of language on 'my' table was so interesting.. with decisions being made about how there should be a consistent approach to and use of language across a school etc.
    - It all highlights for me how much work has gone in to developing into children's learning which results in those Japanese lessons where the children are so seemingly driving the learning.

Learning about CLR from the cycle:

- Planning team: the planning team agreed that an additional session preparing the lesson proposal was really needed. In future we will look to include three sessions after the session sharing research, readings etc. The planning team would have benefited from a meeting after the live-lesson, discussion and Koshi to discuss their reflections on the experience.
- Teacher: teachers need to adjust to the context of teaching with many other people not only in the room but in amongst the children. It was not possible to behave as usual, for example observing across the class to see which pairs were engaged in talk before deciding who to target because adults were blocking the view. This had not been fully appreciated until the event.
- Observers: it would benefit observers not in the planning team to have the opportunity to talk through the mathematics in the research focus in order to gain a shared understanding in advance of the lesson. This would mean they are better prepared for observing and noticing; we make this part of future CLR cycles. Observers were provided with the lesson proposal in advance and additional notes about how to behave (see below).

Notes for Observers

Collaborative Lesson Research 7th May 2019:

Lesson Information

PLEASE DISABLE PHONES TAKEN INTO THE LESSON: NO PHONE SHOULD MAKE A SOUND DURING THE LESSON AND SCHOOL POLICY IS FOR PERSONAL PHONE CALLS TO BE TAKEN ONLY INSIDE OF OFFICE SPACES.

Live lesson: 2:00 – 2:35 (approximately)

During the live lesson you should not interfere with the flow of the lesson at any point.

In order to achieve this there are **THREE RULES** which we all **MUST** stick to:

- **Do not speak to the children (at any point)**
- **Do not speak to the adults (including each other) at any point**
- **Do walk around and look at what the children are doing when they are working but when the teacher is speaking to the class 'dissolve to the sides'**.

Please use your lesson flow to capture observations of what the children say and do. You are encouraged to take photos of children's work (NOT children) and work on the board to support your written observations and to note the time of your individual observations in order to support the post-lesson discussion.

You have been assigned to a table as your focus table where we want you to really notice what these children are doing – you are still able to walk around and look at what other children are doing but by assigning you a table we ensure all the children are attended to and observed. The tables are shown on the next page and are assigned as follows:

Table 1	Table 2	Table 3	Table 4
Laura Nethercott*	Amanda Ballard*	Claire Paton*	Matt Fullerton*
Stuart Boon*	Helen Smith*	John Mortimer*	Bridget Phillips*
Jess Holme	Kayleigh Jeffery	Beth Eckford	Katya O'Neil
Taro Fujita	Andy Tynemouth	Simon Bissett	Rebecca Symons
Matt Pennington	Sam Chapman	Pete Griffin	Tracy Bolton
Helen Edginton	Janine Blinko	Belle Cottingham	Carolyn Wreghitt

**indicates planning team member*

Post-lesson discussion: 2:35 – 3:35 approximately

Please take the opportunity to reflect individually whilst the cameras are being adjusted. Stefanie Burke will manage the post-lesson discussion.

Koshi 3:35 – 4:00 approximately

Matt Lewis will provide further thinking for us all based on the proposal, live lesson and post-lesson discussion.

PLEASE DO NOT LEAVE BEFORE 4:00 UNLESS THERE IS AN EMERGENCY.

We would like you all to complete a reflection and feedback form, following the experience. The link to this is as follows…Thank you for being part of this PD experience.

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*Ruth Trundley and Stefanie Burke are maths advisers with Babcock LDP; this cycle of CLR was part of a research project with six schools. The planning team included selected teachers involved in the research project from five of these schools: The Castle Primary School (Tiverton), Ladysmith Infant and Nursery School (Exeter), Ilfracombe Infants and Nursery School (Ilfracombe), Pilton Infants' School (Barnstaple) and Sticklepath Community School (Barnstaple).